STRUCTURE OF THE FLORETION GROUP

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Abstract. Some characterization of Dement's Floretion Group of order 32, isomorphous to $(C_2 \times D_8) \wedge C_2$, number 49 in GAP's small group library, is given in terms of conjugancy classes, normal subgroups, and relations to the Quaternion Group Q_8.

1. Scope

Dement defines a vector space of floretions [5] with unit vectors taken as the elements of a non-abelian group of order 32 (the Floretion Group) that is defined based on a "right-" and a "left-handed" instance of the Quaternion Group and a commutative multiplication between hybrid elements that are of mixed origin with respect to these sub-groups. It is group 49 in the Besche–Eick table [2, 1, 6] and will henceforth be baptized G_{32}^{49}. It is group 32/42 in the Thomas-Wood enumeration [11], $\Gamma_5 a_1$ in [8], and apparently this index carries over as 32.42 in [7] and $G42$ in [14]. It might also appear as T22, a transitive group of order 8, in a work on permutation groups [3, Table 8A].

In the following we shall ignore that Dement's associated algebra (over the ring/field of $\mathbb{Q}$, so far, see [12] for an application over $\mathbb{C}$) reduces the vector space to 16 dimensions by annihilating a negative sign that is part of the group element's name (see below) with the multiplier -1 of the ring. So with respect of the group properties that will be discussed below, these signs are just an aid to memorization of the Cayley table. The situation is equivalent to the Quaternion Group with its 8 elements and 4 base vectors.

2. Basic Definitions

2.1. **Multiplication Table, Cycles.** The 32 elements of the group ($g_1 = \overleftrightarrow{ee}$ the unit element) are given descriptive names in Table 1. The multiplication table of the Floretion group is reprinted in Tabs. 2–5. The anti-symmetric upper left corner of Table 2 shows a copy of the non-Abelian Quaternion Group with base elements $\overleftrightarrow{ee}$, $\overleftarrow{i}$, $\overleftarrow{j}$, and $\overleftarrow{k}$ in g_1 to g_8 and the standard products

$$\overleftarrow{i}^2 = \overleftarrow{j}^2 = \overleftarrow{k}^2 = -\overleftrightarrow{ee}; \quad (-\overleftrightarrow{ee})^2 = \overleftrightarrow{ee}; \quad \overleftarrow{i}\overleftarrow{j} = \overleftarrow{k}; \quad \overleftarrow{j}\overleftarrow{k} = \overleftarrow{i}; \ldots \tag{1}$$

Another copy with reversal of the sense of the arrows (the opposite "handedness") is with $\overleftrightarrow{ee}$, $\overrightarrow{i}$, $\overrightarrow{j}$, $\overrightarrow{k}$, that is, g_1, g_5 and g_9 to g_{14}. The products of elements with mixed senses of the arrows are defined from there assuming that the elements of

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TABLE 1. Indices of the group elements, their names, and orders.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\overleftrightarrow{ee}$	$\overleftarrow{i}$	$\overleftarrow{j}$	$\overleftarrow{k}$	$-\overleftrightarrow{ee}$	$-\overleftarrow{i}$	$-\overleftarrow{j}$	$-\overleftarrow{k}$	$\overrightarrow{i}$	$\overrightarrow{j}$	$\overrightarrow{k}$	$-\overrightarrow{i}$	$-\overrightarrow{j}$	$-\overrightarrow{k}$	$\overleftrightarrow{ii}$	$\overleftrightarrow{jj}$
1	4	4	4	2	4	4	4	4	4	4	4	4	4	2	2

17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
$\overleftrightarrow{kk}$	$\overleftrightarrow{ij}$	$\overleftrightarrow{ik}$	$\overleftrightarrow{ji}$	$\overleftrightarrow{jk}$	$\overleftrightarrow{ki}$	$\overleftrightarrow{kj}$	$-\overleftrightarrow{ii}$	$-\overleftrightarrow{jj}$	$-\overleftrightarrow{kk}$	$-\overleftrightarrow{ij}$	$-\overleftrightarrow{ik}$	$-\overleftrightarrow{ji}$	$-\overleftrightarrow{jk}$	$-\overleftrightarrow{ki}$	$-\overleftrightarrow{kj}$
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

this mixed origin commute and that the algebraic product of the signs of the factors carries over to the name of the product:

(2)
$$\overleftarrow{i}\,\overrightarrow{i} = \overrightarrow{i}\,\overleftarrow{i} = \overleftrightarrow{ii}\,; \quad \overleftarrow{i}\,\overrightarrow{j} = \overrightarrow{j}\,\overleftarrow{i} = \overleftrightarrow{ij}\,; \quad \overleftarrow{j}\,\overrightarrow{i} = \overrightarrow{i}\,\overleftarrow{j} = \overleftrightarrow{ji}\,; \quad \overleftarrow{i}\,(-\overrightarrow{j}\,) = -\overleftrightarrow{ij}\,; \ldots$$

The products of elements where one or both factors are already of the mixed type are then defined enforcing the associative rule.

Remark 1. *The ordering of the elements from 1 to 32 is entirely my own and others are in use. My principle of ordering is to start with the quaternion base, then to complete the quaternion group in a block, then to follow up with the second quaternion block, then moving on to the mixed products.*

The orders of the elements are listed in Table 1, and a number of small cyclic subgroups of order 2 or 4 is defined by the cycles. The cycle structure in Fig. 1 contains loops with g_1 to g_{14} of the two quaternion groups in the upper part, and the remaining "mixed" elements g_{15} to g_{32} in the lower part.

2.2. **Conjugacy Classes.** The 17 conjugacy classes are $\mathcal{C}_1 = \{g_1\}$, $\mathcal{C}_2 = \{g_2, g_6\}$, $\mathcal{C}_3 = \{g_3, g_7\}$, $\mathcal{C}_4 = \{g_4, g_8\}$, $\mathcal{C}_5 = \{g_5\}$, $\mathcal{C}_9 = \{g_9, g_{12}\}$, $\mathcal{C}_{10} = \{g_{10}, g_{13}\}$, $\mathcal{C}_{11} = \{g_{11}, g_{14}\}$, $\mathcal{C}_{15} = \{g_{15}, g_{24}\}$, $\mathcal{C}_{16} = \{g_{16}, g_{25}\}$, $\mathcal{C}_{17} = \{g_{17}, g_{26}\}$, $\mathcal{C}_{18} = \{g_{18}, g_{27}\}$, $\mathcal{C}_{19} = \{g_{19}, g_{28}\}$, $\mathcal{C}_{20} = \{g_{20}, g_{29}\}$, $\mathcal{C}_{21} = \{g_{21}, g_{30}\}$, $\mathcal{C}_{22} = \{g_{22}, g_{31}\}$, and $\mathcal{C}_{23} = \{g_{23}, g_{32}\}$, indexed by the smallest index of their group members. So with the exception of $\overleftrightarrow{ee}$ and $-\overleftrightarrow{ee}$ which form classes of their own, each two elements with the two signed versions of otherwise the same name build a class.

The class multiplication coefficients are

$$\mathcal{C}_1\mathcal{C}_j = n_j\mathcal{C}_j, \quad , \tag{3}$$
$$\mathcal{C}_5\mathcal{C}_j = n_j\mathcal{C}_j, \quad , \tag{4}$$
$$\mathcal{C}_i\mathcal{C}_i = 2\mathcal{C}_1 + 2\mathcal{C}_5, \quad (i \neq 1, 5), \tag{5}$$
$$\mathcal{C}_i\mathcal{C}_j = 4\mathcal{C}_k, \quad (i \neq j, \quad i, j \neq 1, 5), \tag{6}$$

where n_c denotes the number of elements in class $\mathcal{C}_c$, and where the index k follows immediately from the group multiplication table by multiplication of two representatives of $\mathcal{C}_i$ and $\mathcal{C}_j$.

Remark 2. *A cyclic index relation is valid for the cases of (6): if $\mathcal{C}_i\mathcal{C}_j = 4\mathcal{C}_k$, then also $\mathcal{C}_i\mathcal{C}_k = 4\mathcal{C}_j$. (The proof follows from considering a representative $g_ig_j = g_k$, and multiplying from the left with g_i^{-1} to give $g_j = g_i^{-1}g_k$. Note that the inverses are $g_i^{-1} = g_i$ for $i = 1, 5$ or $i \geq 15$, and flip the sign in the name in all remaining cases. Anyway, g_i^{-1} is in the same class as g_i, so $g_j = g_i^{-1}g_k$ involves three members of the same three classes as $g_ig_j = g_k$.)*

Table 2. Multiplication Table, upper left corner

FLORETION GROUP

		$\overleftrightarrow{ee}$	$\overleftarrow{i}$	$\overleftarrow{j}$	$\overleftarrow{k}$	$-\overleftrightarrow{ee}$	$-\overleftarrow{i}$	$-\overleftarrow{j}$	$-\overleftarrow{k}$	$\overrightarrow{i}$	$\overrightarrow{j}$	$\overrightarrow{k}$	$-\overrightarrow{i}$	$-\overrightarrow{j}$	$-\overrightarrow{k}$	$\overleftrightarrow{ii}$	$\overleftrightarrow{jj}$
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\overleftrightarrow{ee}$	1	$\overleftrightarrow{ee}$	$\overleftarrow{i}$	$\overleftarrow{j}$	$\overleftarrow{k}$	$-\overleftrightarrow{ee}$	$-\overleftarrow{i}$	$-\overleftarrow{j}$	$-\overleftarrow{k}$	$\overrightarrow{i}$	$\overrightarrow{j}$	$\overrightarrow{k}$	$-\overrightarrow{i}$	$-\overrightarrow{j}$	$-\overrightarrow{k}$	$\overleftrightarrow{ii}$	$\overleftrightarrow{jj}$
$\overleftarrow{i}$	2	$\overleftarrow{i}$	$-\overleftrightarrow{ee}$	$\overleftarrow{k}$	$-\overleftarrow{j}$	$-\overleftarrow{i}$	$\overleftrightarrow{ee}$	$-\overleftarrow{k}$	$\overleftarrow{j}$	$\overleftrightarrow{ii}$	$\overleftrightarrow{ij}$	$\overleftrightarrow{ik}$	$-\overleftrightarrow{ii}$	$-\overleftrightarrow{ij}$	$-\overleftrightarrow{ik}$	$-\overrightarrow{i}$	$\overleftrightarrow{kj}$
$\overleftarrow{j}$	3	$\overleftarrow{j}$	$-\overleftarrow{k}$	$-\overleftrightarrow{ee}$	$\overleftarrow{i}$	$-\overleftarrow{j}$	$\overleftarrow{k}$	$\overleftrightarrow{ee}$	$-\overleftarrow{i}$	$\overleftrightarrow{ji}$	$\overleftrightarrow{jj}$	$\overleftrightarrow{jk}$	$-\overleftrightarrow{ji}$	$-\overleftrightarrow{jj}$	$-\overleftrightarrow{jk}$	$-\overleftrightarrow{ki}$	$-\overrightarrow{j}$
$\overleftarrow{k}$	4	$\overleftarrow{k}$	$\overleftarrow{j}$	$-\overleftarrow{i}$	$-\overleftrightarrow{ee}$	$-\overleftarrow{k}$	$-\overleftarrow{j}$	$\overleftarrow{i}$	$\overleftrightarrow{ee}$	$\overleftrightarrow{ki}$	$\overleftrightarrow{kj}$	$\overleftrightarrow{kk}$	$-\overleftrightarrow{ki}$	$-\overleftrightarrow{kj}$	$-\overleftrightarrow{kk}$	$\overleftrightarrow{ji}$	$-\overleftrightarrow{ij}$
$-\overleftrightarrow{ee}$	5	$-\overleftrightarrow{ee}$	$-\overleftarrow{i}$	$-\overleftarrow{j}$	$-\overleftarrow{k}$	$\overleftrightarrow{ee}$	$\overleftarrow{i}$	$\overleftarrow{j}$	$\overleftarrow{k}$	$-\overrightarrow{i}$	$-\overrightarrow{j}$	$-\overrightarrow{k}$	$\overrightarrow{i}$	$\overrightarrow{j}$	$\overrightarrow{k}$	$-\overleftrightarrow{ii}$	$-\overleftrightarrow{jj}$
$-\overleftarrow{i}$	6	$-\overleftarrow{i}$	$\overleftrightarrow{ee}$	$-\overleftarrow{k}$	$\overleftarrow{j}$	$\overleftarrow{i}$	$-\overleftrightarrow{ee}$	$\overleftarrow{k}$	$-\overleftarrow{j}$	$-\overleftrightarrow{ii}$	$-\overleftrightarrow{ij}$	$-\overleftrightarrow{ik}$	$\overleftrightarrow{ii}$	$\overleftrightarrow{ij}$	$\overleftrightarrow{ik}$	$\overrightarrow{i}$	$-\overleftrightarrow{kj}$
$-\overleftarrow{j}$	7	$-\overleftarrow{j}$	$\overleftarrow{k}$	$\overleftrightarrow{ee}$	$-\overleftarrow{i}$	$\overleftarrow{j}$	$-\overleftarrow{k}$	$-\overleftrightarrow{ee}$	$\overleftarrow{i}$	$-\overleftrightarrow{ji}$	$-\overleftrightarrow{jj}$	$-\overleftrightarrow{jk}$	$\overleftrightarrow{ji}$	$\overleftrightarrow{jj}$	$\overleftrightarrow{jk}$	$\overleftrightarrow{ki}$	$\overrightarrow{j}$
$-\overleftarrow{k}$	8	$-\overleftarrow{k}$	$-\overleftarrow{j}$	$\overleftarrow{i}$	$\overleftrightarrow{ee}$	$\overleftarrow{k}$	$\overleftarrow{j}$	$-\overleftarrow{i}$	$-\overleftrightarrow{ee}$	$-\overleftrightarrow{ki}$	$-\overleftrightarrow{kj}$	$-\overleftrightarrow{kk}$	$\overleftrightarrow{ki}$	$\overleftrightarrow{kj}$	$\overleftrightarrow{kk}$	$-\overleftrightarrow{ji}$	$\overleftrightarrow{ij}$
$\overrightarrow{i}$	9	$\overrightarrow{i}$	$\overleftrightarrow{ii}$	$\overleftrightarrow{ji}$	$\overleftrightarrow{ki}$	$-\overrightarrow{i}$	$-\overleftrightarrow{ii}$	$-\overleftrightarrow{ji}$	$-\overleftrightarrow{ki}$	$-\overleftrightarrow{ee}$	$\overrightarrow{k}$	$-\overrightarrow{j}$	$\overleftrightarrow{ee}$	$-\overrightarrow{k}$	$\overrightarrow{j}$	$-\overleftarrow{i}$	$\overleftrightarrow{jk}$
$\overrightarrow{j}$	10	$\overrightarrow{j}$	$\overleftrightarrow{ij}$	$\overleftrightarrow{jj}$	$\overleftrightarrow{kj}$	$-\overrightarrow{j}$	$-\overleftrightarrow{ij}$	$-\overleftrightarrow{jj}$	$-\overleftrightarrow{kj}$	$-\overrightarrow{k}$	$-\overleftrightarrow{ee}$	$\overrightarrow{i}$	$\overrightarrow{k}$	$\overleftrightarrow{ee}$	$-\overrightarrow{i}$	$-\overleftrightarrow{ik}$	$-\overleftarrow{j}$
$\overrightarrow{k}$	11	$\overrightarrow{k}$	$\overleftrightarrow{ik}$	$\overleftrightarrow{jk}$	$\overleftrightarrow{kk}$	$-\overrightarrow{k}$	$-\overleftrightarrow{ik}$	$-\overleftrightarrow{jk}$	$-\overleftrightarrow{kk}$	$\overrightarrow{j}$	$-\overrightarrow{i}$	$-\overleftrightarrow{ee}$	$-\overrightarrow{j}$	$\overrightarrow{i}$	$\overleftrightarrow{ee}$	$\overleftrightarrow{ij}$	$-\overleftrightarrow{ji}$
$-\overrightarrow{i}$	12	$-\overrightarrow{i}$	$-\overleftrightarrow{ii}$	$-\overleftrightarrow{ji}$	$-\overleftrightarrow{ki}$	$\overrightarrow{i}$	$\overleftrightarrow{ii}$	$\overleftrightarrow{ji}$	$\overleftrightarrow{ki}$	$\overleftrightarrow{ee}$	$-\overrightarrow{k}$	$\overrightarrow{j}$	$-\overleftrightarrow{ee}$	$\overrightarrow{k}$	$-\overrightarrow{j}$	$\overleftarrow{i}$	$-\overleftrightarrow{jk}$
$-\overrightarrow{j}$	13	$-\overrightarrow{j}$	$-\overleftrightarrow{ij}$	$-\overleftrightarrow{jj}$	$-\overleftrightarrow{kj}$	$\overrightarrow{j}$	$\overleftrightarrow{ij}$	$\overleftrightarrow{jj}$	$\overleftrightarrow{kj}$	$\overrightarrow{k}$	$\overleftrightarrow{ee}$	$-\overrightarrow{i}$	$-\overrightarrow{k}$	$-\overleftrightarrow{ee}$	$\overrightarrow{i}$	$\overleftrightarrow{ik}$	$\overleftarrow{j}$
$-\overrightarrow{k}$	14	$-\overrightarrow{k}$	$-\overleftrightarrow{ik}$	$-\overleftrightarrow{jk}$	$-\overleftrightarrow{kk}$	$\overrightarrow{k}$	$\overleftrightarrow{ik}$	$\overleftrightarrow{jk}$	$\overleftrightarrow{kk}$	$-\overrightarrow{j}$	$\overrightarrow{i}$	$\overleftrightarrow{ee}$	$\overrightarrow{j}$	$-\overrightarrow{i}$	$-\overleftrightarrow{ee}$	$-\overleftrightarrow{ij}$	$\overleftrightarrow{ji}$
$\overleftrightarrow{ii}$	15	$\overleftrightarrow{ii}$	$-\overrightarrow{i}$	$\overleftrightarrow{ki}$	$-\overleftrightarrow{ji}$	$-\overleftrightarrow{ii}$	$\overrightarrow{i}$	$-\overleftrightarrow{ki}$	$\overleftrightarrow{ji}$	$-\overleftarrow{i}$	$\overleftrightarrow{ik}$	$-\overleftrightarrow{ij}$	$\overleftarrow{i}$	$-\overleftrightarrow{ik}$	$\overleftrightarrow{ij}$	$\overleftrightarrow{ee}$	$\overleftrightarrow{kk}$
$\overleftrightarrow{jj}$	16	$\overleftrightarrow{jj}$	$-\overleftrightarrow{kj}$	$-\overrightarrow{j}$	$\overleftrightarrow{ij}$	$-\overleftrightarrow{jj}$	$\overleftrightarrow{kj}$	$\overrightarrow{j}$	$-\overleftrightarrow{ij}$	$-\overleftrightarrow{jk}$	$-\overleftarrow{j}$	$\overleftrightarrow{ji}$	$\overleftrightarrow{jk}$	$\overleftarrow{j}$	$-\overleftrightarrow{ji}$	$\overleftrightarrow{kk}$	$\overleftrightarrow{ee}$

Table 3. Multiplication Table, upper right corner

		$\overleftrightarrow{kk}$	$\overleftrightarrow{ij}$	$\overleftrightarrow{ik}$	$\overleftrightarrow{ji}$	$\overleftrightarrow{jk}$	$\overleftrightarrow{ki}$	$\overleftrightarrow{kj}$	$-\overleftrightarrow{ii}$	$-\overleftrightarrow{jj}$	$-\overleftrightarrow{kk}$	$-\overleftrightarrow{ij}$	$-\overleftrightarrow{ik}$	$-\overleftrightarrow{ji}$	$-\overleftrightarrow{jk}$	$-\overleftrightarrow{ki}$	$-\overleftrightarrow{kj}$
		17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
$\overleftrightarrow{ee}$	1	$\overleftrightarrow{kk}$	$\overleftrightarrow{ij}$	$\overleftrightarrow{ik}$	$\overleftrightarrow{ji}$	$\overleftrightarrow{jk}$	$\overleftrightarrow{ki}$	$\overleftrightarrow{kj}$	$-\overleftrightarrow{ii}$	$-\overleftrightarrow{jj}$	$-\overleftrightarrow{kk}$	$-\overleftrightarrow{ij}$	$-\overleftrightarrow{ik}$	$-\overleftrightarrow{ji}$	$-\overleftrightarrow{jk}$	$-\overleftrightarrow{ki}$	$-\overleftrightarrow{kj}$
$\overleftarrow{i}$	2	$-\overleftrightarrow{jk}$	$-\overrightarrow{j}$	$-\overrightarrow{k}$	$\overleftrightarrow{ki}$	$\overleftrightarrow{kk}$	$-\overleftrightarrow{ji}$	$-\overleftrightarrow{jj}$	$\overrightarrow{i}$	$-\overleftrightarrow{kj}$	$\overleftrightarrow{jk}$	$\overrightarrow{j}$	$\overrightarrow{k}$	$-\overleftrightarrow{ki}$	$-\overleftrightarrow{kk}$	$\overleftrightarrow{ji}$	$\overleftrightarrow{jj}$
$\overleftarrow{j}$	3	$\overleftrightarrow{ik}$	$-\overleftrightarrow{kj}$	$-\overleftrightarrow{kk}$	$-\overrightarrow{i}$	$-\overrightarrow{k}$	$\overleftrightarrow{ii}$	$\overleftrightarrow{ij}$	$\overleftrightarrow{ki}$	$\overrightarrow{j}$	$-\overleftrightarrow{ik}$	$\overleftrightarrow{kj}$	$\overleftrightarrow{kk}$	$\overrightarrow{i}$	$\overrightarrow{k}$	$-\overleftrightarrow{ii}$	$-\overleftrightarrow{ij}$
$\overleftarrow{k}$	4	$-\overrightarrow{k}$	$\overleftrightarrow{jj}$	$\overleftrightarrow{jk}$	$-\overleftrightarrow{ii}$	$-\overleftrightarrow{ik}$	$-\overrightarrow{i}$	$-\overrightarrow{j}$	$-\overleftrightarrow{ji}$	$\overleftrightarrow{ij}$	$\overrightarrow{k}$	$-\overleftrightarrow{jj}$	$-\overleftrightarrow{jk}$	$\overleftrightarrow{ii}$	$\overleftrightarrow{ik}$	$\overrightarrow{i}$	$\overrightarrow{j}$
$-\overleftrightarrow{ee}$	5	$-\overleftrightarrow{kk}$	$-\overleftrightarrow{ij}$	$-\overleftrightarrow{ik}$	$-\overleftrightarrow{ji}$	$-\overleftrightarrow{jk}$	$-\overleftrightarrow{ki}$	$-\overleftrightarrow{kj}$	$\overleftrightarrow{ii}$	$\overleftrightarrow{jj}$	$\overleftrightarrow{kk}$	$\overleftrightarrow{ij}$	$\overleftrightarrow{ik}$	$\overleftrightarrow{ji}$	$\overleftrightarrow{jk}$	$\overleftrightarrow{ki}$	$\overleftrightarrow{kj}$
$-\overleftarrow{i}$	6	$\overleftrightarrow{jk}$	$\overrightarrow{j}$	$\overrightarrow{k}$	$-\overleftrightarrow{ki}$	$-\overleftrightarrow{kk}$	$\overleftrightarrow{ji}$	$\overleftrightarrow{jj}$	$-\overrightarrow{i}$	$\overleftrightarrow{kj}$	$-\overleftrightarrow{jk}$	$-\overrightarrow{j}$	$-\overrightarrow{k}$	$\overleftrightarrow{ki}$	$\overleftrightarrow{kk}$	$-\overleftrightarrow{ji}$	$-\overleftrightarrow{jj}$
$-\overleftarrow{j}$	7	$-\overleftrightarrow{ik}$	$\overleftrightarrow{kj}$	$\overleftrightarrow{kk}$	$\overrightarrow{i}$	$\overrightarrow{k}$	$-\overleftrightarrow{ii}$	$-\overleftrightarrow{ij}$	$-\overleftrightarrow{ki}$	$-\overrightarrow{j}$	$\overleftrightarrow{ik}$	$-\overleftrightarrow{kj}$	$-\overleftrightarrow{kk}$	$-\overrightarrow{i}$	$-\overrightarrow{k}$	$\overleftrightarrow{ii}$	$\overleftrightarrow{ij}$
$-\overleftarrow{k}$	8	$\overrightarrow{k}$	$-\overleftrightarrow{jj}$	$-\overleftrightarrow{jk}$	$\overleftrightarrow{ii}$	$\overleftrightarrow{ik}$	$\overrightarrow{i}$	$\overrightarrow{j}$	$\overleftrightarrow{ji}$	$-\overleftrightarrow{ij}$	$-\overrightarrow{k}$	$\overleftrightarrow{jj}$	$\overleftrightarrow{jk}$	$-\overleftrightarrow{ii}$	$-\overleftrightarrow{ik}$	$-\overrightarrow{i}$	$-\overrightarrow{j}$
$\overrightarrow{i}$	9	$-\overleftrightarrow{kj}$	$\overleftrightarrow{ik}$	$-\overleftrightarrow{ij}$	$-\overleftarrow{j}$	$-\overleftrightarrow{jj}$	$-\overleftarrow{k}$	$\overleftrightarrow{kk}$	$\overleftarrow{i}$	$-\overleftrightarrow{jk}$	$\overleftrightarrow{kj}$	$-\overleftrightarrow{ik}$	$\overleftrightarrow{ij}$	$\overleftarrow{j}$	$\overleftrightarrow{jj}$	$\overleftarrow{k}$	$-\overleftrightarrow{kk}$
$\overrightarrow{j}$	10	$\overleftrightarrow{ki}$	$-\overleftarrow{i}$	$\overleftrightarrow{ii}$	$-\overleftrightarrow{jk}$	$\overleftrightarrow{ji}$	$-\overleftrightarrow{kk}$	$-\overleftarrow{k}$	$\overleftrightarrow{ik}$	$\overleftarrow{j}$	$-\overleftrightarrow{ki}$	$\overleftarrow{i}$	$-\overleftrightarrow{ii}$	$\overleftrightarrow{jk}$	$-\overleftrightarrow{ji}$	$\overleftrightarrow{kk}$	$\overleftarrow{k}$
$\overrightarrow{k}$	11	$-\overleftarrow{k}$	$-\overleftrightarrow{ii}$	$-\overleftarrow{i}$	$\overleftrightarrow{jj}$	$-\overleftarrow{j}$	$\overleftrightarrow{kj}$	$-\overleftrightarrow{ki}$	$-\overleftrightarrow{ij}$	$\overleftrightarrow{ji}$	$\overleftarrow{k}$	$\overleftrightarrow{ii}$	$\overleftarrow{i}$	$-\overleftrightarrow{jj}$	$\overleftarrow{j}$	$-\overleftrightarrow{kj}$	$\overleftrightarrow{ki}$
$-\overrightarrow{i}$	12	$\overleftrightarrow{kj}$	$-\overleftrightarrow{ik}$	$\overleftrightarrow{ij}$	$\overleftarrow{j}$	$\overleftrightarrow{jj}$	$\overleftarrow{k}$	$-\overleftrightarrow{kk}$	$-\overleftarrow{i}$	$\overleftrightarrow{jk}$	$-\overleftrightarrow{kj}$	$\overleftrightarrow{ik}$	$-\overleftrightarrow{ij}$	$-\overleftarrow{j}$	$-\overleftrightarrow{jj}$	$-\overleftarrow{k}$	$\overleftrightarrow{kk}$
$-\overrightarrow{j}$	13	$-\overleftrightarrow{ki}$	$\overleftarrow{i}$	$-\overleftrightarrow{ii}$	$\overleftrightarrow{jk}$	$-\overleftrightarrow{ji}$	$\overleftrightarrow{kk}$	$\overleftarrow{k}$	$-\overleftrightarrow{ik}$	$-\overleftarrow{j}$	$\overleftrightarrow{ki}$	$-\overleftarrow{i}$	$\overleftrightarrow{ii}$	$-\overleftrightarrow{jk}$	$\overleftrightarrow{ji}$	$-\overleftrightarrow{kk}$	$-\overleftarrow{k}$
$-\overrightarrow{k}$	14	$\overleftarrow{k}$	$\overleftrightarrow{ii}$	$\overleftarrow{i}$	$-\overleftrightarrow{jj}$	$\overleftarrow{j}$	$-\overleftrightarrow{kj}$	$\overleftrightarrow{ki}$	$\overleftrightarrow{ij}$	$-\overleftrightarrow{ji}$	$-\overleftarrow{k}$	$-\overleftrightarrow{ii}$	$-\overleftarrow{i}$	$\overleftrightarrow{jj}$	$-\overleftarrow{j}$	$\overleftrightarrow{kj}$	$-\overleftrightarrow{ki}$
$\overleftrightarrow{ii}$	15	$\overleftrightarrow{jj}$	$-\overrightarrow{k}$	$\overrightarrow{j}$	$-\overleftarrow{k}$	$-\overleftrightarrow{kj}$	$\overleftarrow{j}$	$-\overleftrightarrow{jk}$	$-\overleftrightarrow{ee}$	$-\overleftrightarrow{kk}$	$-\overleftrightarrow{jj}$	$\overrightarrow{k}$	$-\overrightarrow{j}$	$\overleftarrow{k}$	$\overleftrightarrow{kj}$	$-\overleftarrow{j}$	$\overleftrightarrow{jk}$
$\overleftrightarrow{jj}$	16	$\overleftrightarrow{ii}$	$\overleftarrow{k}$	$-\overleftrightarrow{ki}$	$\overrightarrow{k}$	$-\overrightarrow{i}$	$-\overleftrightarrow{ik}$	$-\overleftarrow{i}$	$-\overleftrightarrow{kk}$	$-\overleftrightarrow{ee}$	$-\overleftrightarrow{ii}$	$-\overleftarrow{k}$	$\overleftrightarrow{ki}$	$-\overrightarrow{k}$	$\overrightarrow{i}$	$\overleftrightarrow{ik}$	$\overleftarrow{i}$

Table 4. Multiplication Table, lower left corner

		$\overleftrightarrow{ee}$ 1	$\overleftarrow{i}$ 2	$\overleftarrow{j}$ 3	$\overleftarrow{k}$ 4	$-\overleftrightarrow{ee}$ 5	$-\overleftarrow{i}$ 6	$-\overleftarrow{j}$ 7	$-\overleftarrow{k}$ 8	$\overrightarrow{i}$ 9	$\overrightarrow{j}$ 10	$\overrightarrow{k}$ 11	$-\overrightarrow{i}$ 12	$-\overrightarrow{j}$ 13	$-\overrightarrow{k}$ 14	$\overleftrightarrow{ii}$ 15	$\overleftrightarrow{jj}$ 16
$\overleftrightarrow{kk}$	17	$\overleftrightarrow{kk}$	$\overleftrightarrow{jk}$	$-\overleftrightarrow{ik}$	$-\overrightarrow{k}$	$-\overleftrightarrow{kk}$	$-\overleftrightarrow{jk}$	$\overleftrightarrow{ik}$	$\overrightarrow{k}$	$\overleftrightarrow{kj}$	$-\overleftrightarrow{ki}$	$-\overleftarrow{k}$	$-\overleftrightarrow{kj}$	$\overleftrightarrow{ki}$	$\overleftarrow{k}$	$\overleftrightarrow{jj}$	$\overleftrightarrow{ii}$
$\overleftrightarrow{ij}$	18	$\overleftrightarrow{ij}$	$-\overrightarrow{j}$	$\overleftrightarrow{kj}$	$-\overleftrightarrow{jj}$	$-\overleftrightarrow{ij}$	$\overrightarrow{j}$	$-\overleftrightarrow{kj}$	$\overleftrightarrow{jj}$	$-\overleftrightarrow{ik}$	$-\overleftarrow{i}$	$\overleftrightarrow{ii}$	$\overleftrightarrow{ik}$	$\overleftarrow{i}$	$-\overleftrightarrow{ii}$	$\overrightarrow{k}$	$-\overleftarrow{k}$
$\overleftrightarrow{ik}$	19	$\overleftrightarrow{ik}$	$-\overrightarrow{k}$	$\overleftrightarrow{kk}$	$-\overleftrightarrow{jk}$	$-\overleftrightarrow{ik}$	$\overrightarrow{k}$	$-\overleftrightarrow{kk}$	$\overleftrightarrow{jk}$	$\overleftrightarrow{ij}$	$-\overleftrightarrow{ii}$	$-\overleftarrow{i}$	$-\overleftrightarrow{ij}$	$\overleftrightarrow{ii}$	$\overleftarrow{i}$	$-\overrightarrow{j}$	$-\overleftrightarrow{ki}$
$\overleftrightarrow{ji}$	20	$\overleftrightarrow{ji}$	$-\overleftrightarrow{ki}$	$-\overrightarrow{i}$	$\overleftrightarrow{ii}$	$-\overleftrightarrow{ji}$	$\overleftrightarrow{ki}$	$\overrightarrow{i}$	$-\overleftrightarrow{ii}$	$-\overleftarrow{j}$	$\overleftrightarrow{jk}$	$-\overleftrightarrow{jj}$	$\overleftarrow{j}$	$-\overleftrightarrow{jk}$	$\overleftrightarrow{jj}$	$\overleftarrow{k}$	$-\overrightarrow{k}$
$\overleftrightarrow{jk}$	21	$\overleftrightarrow{jk}$	$-\overleftrightarrow{kk}$	$-\overrightarrow{k}$	$\overleftrightarrow{ik}$	$-\overleftrightarrow{jk}$	$\overleftrightarrow{kk}$	$\overrightarrow{k}$	$-\overleftrightarrow{ik}$	$\overleftrightarrow{jj}$	$-\overleftrightarrow{ji}$	$-\overleftarrow{j}$	$-\overleftrightarrow{jj}$	$\overleftrightarrow{ji}$	$\overleftarrow{j}$	$-\overleftrightarrow{kj}$	$\overrightarrow{i}$
$\overleftrightarrow{ki}$	22	$\overleftrightarrow{ki}$	$\overleftrightarrow{ji}$	$-\overleftrightarrow{ii}$	$-\overrightarrow{i}$	$-\overleftrightarrow{ki}$	$-\overleftrightarrow{ji}$	$\overleftrightarrow{ii}$	$\overrightarrow{i}$	$-\overleftarrow{k}$	$\overleftrightarrow{kk}$	$-\overleftrightarrow{kj}$	$\overleftarrow{k}$	$-\overleftrightarrow{kk}$	$\overleftrightarrow{kj}$	$-\overleftarrow{j}$	$-\overleftrightarrow{ik}$
$\overleftrightarrow{kj}$	23	$\overleftrightarrow{kj}$	$\overleftrightarrow{jj}$	$-\overleftrightarrow{ij}$	$-\overrightarrow{j}$	$-\overleftrightarrow{kj}$	$-\overleftrightarrow{jj}$	$\overleftrightarrow{ij}$	$\overrightarrow{j}$	$-\overleftrightarrow{kk}$	$-\overleftarrow{k}$	$\overleftrightarrow{ki}$	$\overleftrightarrow{kk}$	$\overleftarrow{k}$	$-\overleftrightarrow{ki}$	$-\overleftrightarrow{jk}$	$\overleftarrow{i}$
$-\overleftrightarrow{ii}$	24	$-\overleftrightarrow{ii}$	$\overrightarrow{i}$	$-\overleftrightarrow{ki}$	$\overleftrightarrow{ji}$	$\overleftrightarrow{ii}$	$-\overrightarrow{i}$	$\overleftrightarrow{ki}$	$-\overleftrightarrow{ji}$	$\overleftarrow{i}$	$-\overleftrightarrow{ik}$	$\overleftrightarrow{ij}$	$-\overleftarrow{i}$	$\overleftrightarrow{ik}$	$-\overleftrightarrow{ij}$	$-\overleftrightarrow{ee}$	$-\overleftrightarrow{kk}$
$-\overleftrightarrow{jj}$	25	$-\overleftrightarrow{jj}$	$\overleftrightarrow{kj}$	$\overrightarrow{j}$	$-\overleftrightarrow{ij}$	$\overleftrightarrow{jj}$	$-\overleftrightarrow{kj}$	$-\overrightarrow{j}$	$\overleftrightarrow{ij}$	$\overleftrightarrow{jk}$	$\overleftarrow{j}$	$-\overleftrightarrow{ji}$	$-\overleftrightarrow{jk}$	$-\overleftarrow{j}$	$\overleftrightarrow{ji}$	$-\overleftrightarrow{kk}$	$-\overleftrightarrow{ee}$
$-\overleftrightarrow{kk}$	26	$-\overleftrightarrow{kk}$	$-\overleftrightarrow{jk}$	$\overleftrightarrow{ik}$	$\overrightarrow{k}$	$\overleftrightarrow{kk}$	$\overleftrightarrow{jk}$	$-\overleftrightarrow{ik}$	$-\overrightarrow{k}$	$-\overleftrightarrow{kj}$	$\overleftrightarrow{ki}$	$\overleftarrow{k}$	$\overleftrightarrow{kj}$	$-\overleftrightarrow{ki}$	$-\overleftarrow{k}$	$-\overleftrightarrow{jj}$	$-\overleftrightarrow{ii}$
$-\overleftrightarrow{ij}$	27	$-\overleftrightarrow{ij}$	$\overrightarrow{j}$	$-\overleftrightarrow{kj}$	$\overleftrightarrow{jj}$	$\overleftrightarrow{ij}$	$-\overrightarrow{j}$	$\overleftrightarrow{kj}$	$-\overleftrightarrow{jj}$	$\overleftrightarrow{ik}$	$\overleftarrow{i}$	$-\overleftrightarrow{ii}$	$-\overleftrightarrow{ik}$	$-\overleftarrow{i}$	$\overleftrightarrow{ii}$	$-\overrightarrow{k}$	$\overleftarrow{k}$
$-\overleftrightarrow{ik}$	28	$-\overleftrightarrow{ik}$	$\overrightarrow{k}$	$-\overleftrightarrow{kk}$	$\overleftrightarrow{jk}$	$\overleftrightarrow{ik}$	$-\overrightarrow{k}$	$\overleftrightarrow{kk}$	$-\overleftrightarrow{jk}$	$-\overleftrightarrow{ij}$	$\overleftrightarrow{ii}$	$\overleftarrow{i}$	$\overleftrightarrow{ij}$	$-\overleftrightarrow{ii}$	$-\overleftarrow{i}$	$\overrightarrow{j}$	$\overleftrightarrow{ki}$
$-\overleftrightarrow{ji}$	29	$-\overleftrightarrow{ji}$	$\overleftrightarrow{ki}$	$\overrightarrow{i}$	$-\overleftrightarrow{ii}$	$\overleftrightarrow{ji}$	$-\overleftrightarrow{ki}$	$-\overrightarrow{i}$	$\overleftrightarrow{ii}$	$\overleftarrow{j}$	$-\overleftrightarrow{jk}$	$\overleftrightarrow{jj}$	$-\overleftarrow{j}$	$\overleftrightarrow{jk}$	$-\overleftrightarrow{jj}$	$-\overleftarrow{k}$	$\overrightarrow{k}$
$-\overleftrightarrow{jk}$	30	$-\overleftrightarrow{jk}$	$\overleftrightarrow{kk}$	$\overrightarrow{k}$	$-\overleftrightarrow{ik}$	$\overleftrightarrow{jk}$	$-\overleftrightarrow{kk}$	$-\overrightarrow{k}$	$\overleftrightarrow{ik}$	$-\overleftrightarrow{jj}$	$\overleftrightarrow{ji}$	$\overleftarrow{j}$	$\overleftrightarrow{jj}$	$-\overleftrightarrow{ji}$	$-\overleftarrow{j}$	$\overleftrightarrow{kj}$	$-\overrightarrow{i}$
$-\overleftrightarrow{ki}$	31	$-\overleftrightarrow{ki}$	$-\overleftrightarrow{ji}$	$\overleftrightarrow{ii}$	$\overrightarrow{i}$	$\overleftrightarrow{ki}$	$\overleftrightarrow{ji}$	$-\overleftrightarrow{ii}$	$-\overrightarrow{i}$	$\overleftarrow{k}$	$-\overleftrightarrow{kk}$	$\overleftrightarrow{kj}$	$-\overleftarrow{k}$	$\overleftrightarrow{kk}$	$-\overleftrightarrow{kj}$	$\overleftarrow{j}$	$\overleftrightarrow{ik}$
$-\overleftrightarrow{kj}$	32	$-\overleftrightarrow{kj}$	$-\overleftrightarrow{jj}$	$\overleftrightarrow{ij}$	$\overrightarrow{j}$	$\overleftrightarrow{kj}$	$\overleftrightarrow{jj}$	$-\overleftrightarrow{ij}$	$-\overrightarrow{j}$	$\overleftrightarrow{kk}$	$\overleftarrow{k}$	$-\overleftrightarrow{ki}$	$-\overleftrightarrow{kk}$	$-\overleftarrow{k}$	$\overleftrightarrow{ki}$	$\overleftrightarrow{jk}$	$-\overleftarrow{i}$

TABLE 5. Multiplication Table, lower right corner

		$\overleftrightarrow{kk}$ 17	$\overleftrightarrow{ij}$ 18	$\overleftrightarrow{ik}$ 19	$\overleftrightarrow{ji}$ 20	$\overleftrightarrow{jk}$ 21	$\overleftrightarrow{ki}$ 22	$\overleftrightarrow{kj}$ 23	$-\overleftrightarrow{ii}$ 24	$-\overleftrightarrow{jj}$ 25	$-\overleftrightarrow{kk}$ 26	$-\overleftrightarrow{ij}$ 27	$-\overleftrightarrow{ik}$ 28	$-\overleftrightarrow{ji}$ 29	$-\overleftrightarrow{jk}$ 30	$-\overleftrightarrow{ki}$ 31	$-\overleftrightarrow{kj}$ 32
$\overleftrightarrow{kk}$	17	$\overleftrightarrows{ee}$	$-\overleftrightarrow{ji}$	$-\overleftarrow{j}$	$-\overleftrightarrow{ij}$	$\overleftarrow{i}$	$-\overrightarrow{j}$	$\overrightarrow{i}$	$-\overleftrightarrow{jj}$	$-\overleftrightarrow{ii}$	$-\overleftrightarrows{ee}$	$\overleftrightarrow{ji}$	$\overleftarrow{j}$	$\overleftrightarrow{ij}$	$-\overleftarrow{i}$	$\overrightarrow{j}$	$-\overrightarrow{i}$
$\overleftrightarrow{ij}$	18	$-\overleftrightarrow{ji}$	$\overleftrightarrows{ee}$	$-\overrightarrow{i}$	$-\overleftrightarrow{kk}$	$\overleftrightarrow{ki}$	$\overleftrightarrow{jk}$	$\overleftarrow{j}$	$-\overrightarrow{k}$	$\overleftarrow{k}$	$\overleftrightarrow{ji}$	$-\overleftrightarrows{ee}$	$\overrightarrow{i}$	$\overleftrightarrow{kk}$	$-\overleftrightarrow{ki}$	$-\overleftrightarrow{jk}$	$-\overleftarrow{j}$
$\overleftrightarrow{ik}$	19	$\overleftarrow{j}$	$\overrightarrow{i}$	$\overleftrightarrows{ee}$	$\overleftrightarrow{kj}$	$-\overleftarrow{k}$	$-\overleftrightarrow{jj}$	$\overleftrightarrow{ji}$	$\overrightarrow{j}$	$\overleftrightarrow{ki}$	$-\overleftarrow{j}$	$-\overrightarrow{i}$	$-\overleftrightarrows{ee}$	$-\overleftrightarrow{kj}$	$\overleftarrow{k}$	$\overleftrightarrow{jj}$	$-\overleftrightarrow{ji}$
$\overleftrightarrow{ji}$	20	$-\overleftrightarrow{ij}$	$-\overleftrightarrow{kk}$	$\overleftrightarrow{kj}$	$\overleftrightarrows{ee}$	$\overrightarrow{j}$	$-\overleftarrow{i}$	$\overleftrightarrow{ik}$	$-\overleftarrow{k}$	$\overrightarrow{k}$	$\overleftrightarrow{ij}$	$\overleftrightarrow{kk}$	$-\overleftrightarrow{kj}$	$-\overleftrightarrows{ee}$	$-\overrightarrow{j}$	$\overleftarrow{i}$	$-\overleftrightarrow{ik}$
$\overleftrightarrow{jk}$	21	$-\overleftarrow{i}$	$\overleftrightarrow{ki}$	$\overleftarrow{k}$	$-\overrightarrow{j}$	$\overleftrightarrows{ee}$	$\overleftrightarrow{ij}$	$-\overleftrightarrow{ii}$	$\overleftrightarrow{kj}$	$-\overrightarrow{i}$	$\overleftarrow{i}$	$-\overleftrightarrow{ki}$	$-\overleftarrow{k}$	$\overrightarrow{j}$	$-\overleftrightarrows{ee}$	$-\overleftrightarrow{ij}$	$\overleftrightarrow{ii}$
$\overleftrightarrow{ki}$	22	$\overrightarrow{j}$	$\overleftrightarrow{jk}$	$-\overleftrightarrow{jj}$	$\overleftarrow{i}$	$\overleftrightarrow{ij}$	$\overleftrightarrows{ee}$	$-\overrightarrow{k}$	$\overleftarrow{j}$	$\overleftrightarrow{ik}$	$-\overrightarrow{j}$	$-\overleftrightarrow{jk}$	$\overleftrightarrow{jj}$	$-\overleftarrow{i}$	$-\overleftrightarrow{ij}$	$-\overleftrightarrows{ee}$	$\overrightarrow{k}$
$\overleftrightarrow{kj}$	23	$-\overrightarrow{i}$	$-\overleftarrow{j}$	$\overleftrightarrow{ji}$	$\overleftrightarrow{ik}$	$-\overleftrightarrow{ii}$	$\overrightarrow{k}$	$\overleftrightarrows{ee}$	$\overleftrightarrow{jk}$	$-\overleftarrow{i}$	$\overrightarrow{i}$	$\overleftarrow{j}$	$-\overleftrightarrow{ji}$	$-\overleftrightarrow{ik}$	$\overleftrightarrow{ii}$	$-\overrightarrow{k}$	$-\overleftrightarrows{ee}$
$-\overleftrightarrow{ii}$	24	$-\overleftrightarrow{jj}$	$\overrightarrow{k}$	$-\overrightarrow{j}$	$\overleftarrow{k}$	$\overleftrightarrow{kj}$	$-\overleftarrow{j}$	$\overleftrightarrow{jk}$	$\overleftrightarrows{ee}$	$\overleftrightarrow{kk}$	$\overleftrightarrow{jj}$	$-\overrightarrow{k}$	$\overrightarrow{j}$	$-\overleftarrow{k}$	$-\overleftrightarrow{kj}$	$\overleftarrow{j}$	$-\overleftrightarrow{jk}$
$-\overleftrightarrow{jj}$	25	$-\overleftrightarrow{ii}$	$-\overleftarrow{k}$	$\overleftrightarrow{ki}$	$-\overrightarrow{k}$	$\overrightarrow{i}$	$\overleftrightarrow{ik}$	$\overleftarrow{i}$	$\overleftrightarrow{kk}$	$\overleftrightarrows{ee}$	$\overleftrightarrow{ii}$	$\overleftarrow{k}$	$-\overleftrightarrow{ki}$	$\overrightarrow{k}$	$-\overrightarrow{i}$	$-\overleftrightarrow{ik}$	$-\overleftarrow{i}$
$-\overleftrightarrow{kk}$	26	$-\overleftrightarrows{ee}$	$\overleftrightarrow{ji}$	$\overleftarrow{j}$	$\overleftrightarrow{ij}$	$-\overleftarrow{i}$	$\overrightarrow{j}$	$-\overrightarrow{i}$	$\overleftrightarrow{jj}$	$\overleftrightarrow{ii}$	$\overleftrightarrows{ee}$	$-\overleftrightarrow{ji}$	$-\overleftarrow{j}$	$-\overleftrightarrow{ij}$	$\overleftarrow{i}$	$-\overrightarrow{j}$	$\overrightarrow{i}$
$-\overleftrightarrow{ij}$	27	$\overleftrightarrow{ji}$	$-\overleftrightarrows{ee}$	$\overrightarrow{i}$	$\overleftrightarrow{kk}$	$-\overleftrightarrow{ki}$	$-\overleftrightarrow{jk}$	$-\overleftarrow{j}$	$\overrightarrow{k}$	$-\overleftarrow{k}$	$-\overleftrightarrow{ji}$	$\overleftrightarrows{ee}$	$-\overrightarrow{i}$	$-\overleftrightarrow{kk}$	$\overleftrightarrow{ki}$	$\overleftrightarrow{jk}$	$\overleftarrow{j}$
$-\overleftrightarrow{ik}$	28	$-\overleftarrow{j}$	$-\overrightarrow{i}$	$-\overleftrightarrows{ee}$	$-\overleftrightarrow{kj}$	$\overleftarrow{k}$	$\overleftrightarrow{jj}$	$-\overleftrightarrow{ji}$	$-\overrightarrow{j}$	$-\overleftrightarrow{ki}$	$\overleftarrow{j}$	$\overrightarrow{i}$	$\overleftrightarrows{ee}$	$\overleftrightarrow{kj}$	$-\overleftarrow{k}$	$-\overleftrightarrow{jj}$	$\overleftrightarrow{ji}$
$-\overleftrightarrow{ji}$	29	$\overleftrightarrow{ij}$	$\overleftrightarrow{kk}$	$-\overleftrightarrow{kj}$	$-\overleftrightarrows{ee}$	$-\overrightarrow{j}$	$\overleftarrow{i}$	$-\overleftrightarrow{ik}$	$\overleftarrow{k}$	$-\overrightarrow{k}$	$-\overleftrightarrow{ij}$	$-\overleftrightarrow{kk}$	$\overleftrightarrow{kj}$	$\overleftrightarrows{ee}$	$\overrightarrow{j}$	$-\overleftarrow{i}$	$\overleftrightarrow{ik}$
$-\overleftrightarrow{jk}$	30	$\overleftarrow{i}$	$-\overleftrightarrow{ki}$	$-\overleftarrow{k}$	$\overrightarrow{j}$	$-\overleftrightarrows{ee}$	$-\overleftrightarrow{ij}$	$\overleftrightarrow{ii}$	$-\overleftrightarrow{kj}$	$\overrightarrow{i}$	$-\overleftarrow{i}$	$\overleftrightarrow{ki}$	$\overleftarrow{k}$	$-\overrightarrow{j}$	$\overleftrightarrows{ee}$	$\overleftrightarrow{ij}$	$-\overleftrightarrow{ii}$
$-\overleftrightarrow{ki}$	31	$-\overrightarrow{j}$	$-\overleftrightarrow{jk}$	$\overleftrightarrow{jj}$	$-\overleftarrow{i}$	$-\overleftrightarrow{ij}$	$-\overleftrightarrows{ee}$	$\overrightarrow{k}$	$-\overleftarrow{j}$	$-\overleftrightarrow{ik}$	$\overrightarrow{j}$	$\overleftrightarrow{jk}$	$-\overleftrightarrow{jj}$	$\overleftarrow{i}$	$\overleftrightarrow{ij}$	$\overleftrightarrows{ee}$	$-\overrightarrow{k}$
$-\overleftrightarrow{kj}$	32	$\overrightarrow{i}$	$\overleftarrow{j}$	$-\overleftrightarrow{ji}$	$-\overleftrightarrow{ik}$	$\overleftrightarrow{ii}$	$-\overrightarrow{k}$	$-\overleftrightarrows{ee}$	$-\overleftrightarrow{jk}$	$\overleftarrow{i}$	$-\overrightarrow{i}$	$-\overleftarrow{j}$	$\overleftrightarrow{ji}$	$\overleftrightarrow{ik}$	$-\overleftrightarrow{ii}$	$\overrightarrow{k}$	$\overleftrightarrows{ee}$

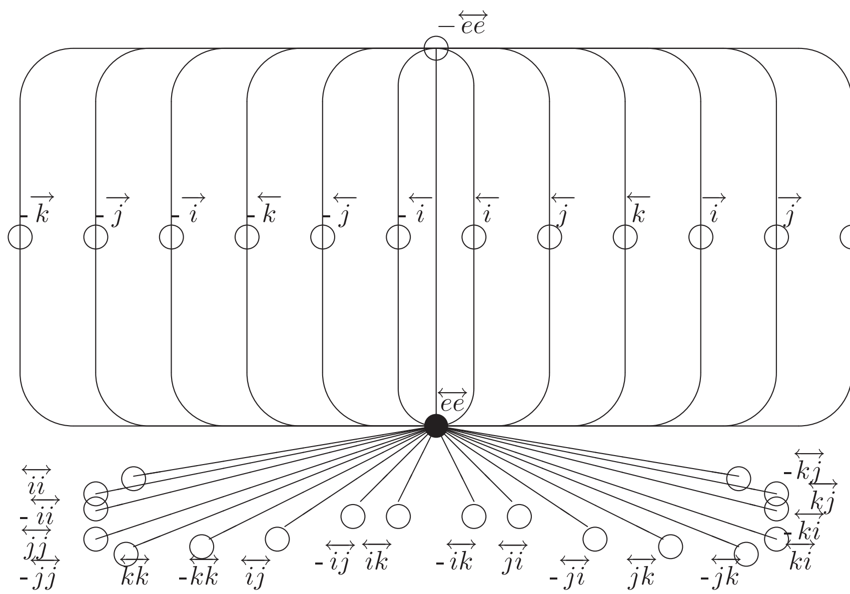

FIGURE 1. Cycle graph of the Floretion Group. In the upper part, the six loops are to be closed switching signs while passing through the nodes g_1 or g_5.

TABLE 6. Characters of the 17 representations.

$\mathcal{C}$	1	2	3	4	5	9	10	11	15	16	17	18	19	20	21	22	23
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	-1	-1	1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1
	1	-1	1	-1	1	1	1	1	-1	1	-1	-1	-1	1	1	-1	-1
	1	-1	-1	1	1	1	-1	-1	-1	1	-1	1	1	-1	1	1	-1
	1	-1	1	-1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	1
	1	1	-1	-1	1	1	-1	-1	1	1	1	-1	-1	-1	1	-1	1
	1	-1	-1	1	1	-1	-1	1	1	1	1	1	-1	1	-1	-1	-1
	1	1	1	1	1	-1	-1	1	-1	-1	1	-1	1	-1	1	-1	-1
	1	-1	1	-1	1	-1	1	-1	1	1	1	-1	1	-1	-1	1	-1
	1	1	1	1	1	1	-1	-1	1	-1	-1	-1	-1	1	-1	1	-1
	1	1	-1	-1	1	-1	1	-1	-1	-1	1	1	-1	1	1	1	-1
	1	1	1	1	1	-1	1	-1	-1	1	-1	1	-1	-1	-1	-1	1
	1	-1	-1	1	1	-1	1	-1	1	-1	-1	-1	1	1	1	-1	1
	1	-1	-1	1	1	1	1	1	-1	-1	1	-1	-1	-1	-1	1	1
	1	1	-1	-1	1	-1	-1	1	-1	1	-1	-1	1	1	-1	1	1
	1	-1	1	-1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	1	1
	4	0	0	0	-4	0	0	0	0	0	0	0	0	0	0	0	0

There are 16 irreducible representations of dimension 1 and 1 with dimension 4 (Table 6).

The inner automorphism group of G_{32}^{49} is $C_2 \times C_2 \times C_2 \times C_2$ [11, Group 16/6].

3. Derived Properties

3.1. Subgroups.

Definition 1. *(Small Groups)* G_d^j *denotes the* j*-th abstract group of order* d *of the small group library* [6].

Definition 2. *(Normal Subgroups of* G_{32}^{49}*)* $H_d^{i,j,k,\cdots}$ *denotes a normal (invariant) subgroup of order* d *of the Floretion Group* G_{32}^{49} *containing all elements of the classes* $\mathcal{C}_1$, $\mathcal{C}_5$ $\mathcal{C}_i$, $\mathcal{C}_j$, $\mathcal{C}_k$ *etc., that is,* $\mathcal{C}_1$, $\mathcal{C}_5$ *and the classes indicated by the upper indices.*

Definition 3. *(Other Subgroups of* G_{32}^{49}*)* $M_d^{i,j,k,\cdots}$ *denotes a not normal subgroup of order* d *of the Floretion Group* G_{32}^{49} *containing the elements* g_1, g_i, g_j, g_k *etc., that is,* g_1 *and the elements indicated by the upper indices.*

Besides the improper subgroup $M_1 = \{g_1\}$, we identify the following subgroups from Figure 1 (all cyclic and therefore Abelian):

- The commutator sup-group $H_2 = \{g_1, g_5\}$ (set of all commutators $g^{-1}h^{-1}gh$ of pair-wise elements g and h) contains the two elements $\overleftrightarrow{ee}$ and $-\overleftrightarrow{ee}$, isomorphous to $C_2 = G_2^1$, the cyclic group of order 2.
- 6 normal subgroups of order 4 (index 8) given by $H_4^i = \{g_1, g_5, g_i, g_i^{-1}\} \leftrightarrow C_4 = G_4^1$, $i \in \{2, 3, 4, 9, 10, 11\}$, containing the cycle generated by any of the 6 elements of the two Quaternion groups.
- 18 subgroups of order 2 (index 16) given by $M_2^i \leftrightarrow C_2$, $i \geq 15$, containing the unit element and one of the "mixed" elements. None of them is a normal subgroup of G_{32}^{49}.

By merging the elements of pairs of any of these subgroups we generate 62 more subgroups:

- The two normal, non-Abelian Quaternion subgroups of order 8 isomorphous to $Q_8 = G_8^4$, $H_8^{2,3,4} = \mathcal{C}_1 + \mathcal{C}_5 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4$ and $H_8^{9,10,11} = \mathcal{C}_1 + \mathcal{C}_5 + \mathcal{C}_9 + \mathcal{C}_{10} + \mathcal{C}_{11}$.
- 9 normal Abelian subgroups of order 8 by merging two of the aforementioned subgroups of order 4 with different senses of the arrows, and adding the complementary two "mixed" elements: $H_8^{2,9,15}$, $H_8^{2,10,18}$, $H_8^{2,11,19}$, $H_8^{3,9,20}$, $H_8^{3,10,16}$, $H_8^{3,11,21}$, $H_8^{4,9,22}$, $H_8^{4,10,23}$, or $H_8^{4,11,17}$, isomorphous to $C_4 \times C_2 = G_8^2$.
- 18 normal non-Abelian subgroups of order 8 by merging one of cyclic subgroups of order 4 with two cyclic subgroups of order 2: $H_8^{4,15,20}$, $H_8^{4,16,18}$, $H_8^{4,19,21}$, $H_8^{9,16,21}$, $H_8^{9,17,23}$, $H_8^{9,18,19}$, $H_8^{10,15,19}$, $H_8^{10,17,22}$, $H_8^{10,20,21}$, $H_8^{11,15,18}$, $H_8^{11,16,20}$, $H_8^{11,22,23}$, $H_8^{3,18,23}$, $H_8^{2,16,23}$, $H_8^{2,17,21}$, $H_8^{2,20,22}$, $H_8^{3,15,22}$, $H_8^{3,17,19}$, isomorphous to $D_8 = G_8^3$.
- 9 normal Abelian subgroups of order 4 by merging pairs within the aforementioned 18 subgroups of order 2 with the rule that both sign variants are combined: $H_4^i \leftrightarrow C_2 \times C_2 = G_4^2$, $i \geq 15$.
- 24 Abelian subgroups combining the unit with three "mixed" elements: $M_4^{15,25,26}$, $M_4^{18,20,26}$, $M_4^{15,23,30}$, $M_4^{18,21,22}$, $M_4^{20,28,32}$, $M_4^{17,18,29}$, $M_4^{24,30,32}$, $M_4^{21,23,24}$, $M_4^{16,19,31}$, $M_4^{19,20,23}$, $M_4^{22,27,30}$, $M_4^{26,27,29}$, $M_4^{15,16,17}$, $M_4^{17,20,27}$, $M_4^{19,22,25}$, $M_4^{21,27,31}$, $M_4^{15,21,32}$, $M_4^{16,22,28}$, $M_4^{25,28,31}$, $M_4^{19,29,32}$, $M_4^{18,30,31}$, $M_4^{17,24,25}$, $M_4^{16,24,26}$, or $M_4^{23,28,29}$, isomorphous to $C_2 \times C_2$.

Remark 3. *Proposition 2.5 in* [4] *concerning the number of subgroups of order 4 is not correct because some of the 24 subgroups mentioned above are missed.*

After a second round of mergers we find more normal subgroups [13]

- non-Abelian of order 16: $H_{16}^{2,3,4,11,17,19,21}$, $H_{16}^{2,3,4,9,15,20,22}$, $H_{16}^{2,3,4,10,16,18,23}$, $H_{16}^{2,9,10,11,15,18,19}$, $H_{16}^{3,9,10,11,16,20,21}$, $H_{16}^{4,9,10,11,17,22,23}$, isomorphous to $(C_4 \times C_2) \wedge C_2 = G_{16}^{13}$.
- non-Abelian of order 16: $H_{16}^{2,10,17,18,20,21,22}$, $H_{16}^{2,11,16,19,20,22,23}$, $H_{16}^{2,9,15,16,17,21,23}$, $H_{16}^{3,11,15,18,21,22,23}$, $H_{16}^{3,9,17,18,19,20,23}$, $H_{16}^{3,10,15,16,17,19,22}$, $H_{16}^{4,9,16,18,19,21,22}$, $H_{16}^{4,10,15,19,20,21,23}$, $H_{16}^{4,11,15,16,17,18,20}$, isomorphous to $C_2 \times D_8 = G_{16}^{11}$.
- and Abelian of order 8: $H_8^{18,21,22}$, $H_8^{19,20,23}$, $H_8^{17,18,20}$, $H_8^{16,19,22}$, $H_8^{15,21,23}$, $H_8^{15,16,17}$, isomorphous to $C_2 \times C_2 \times C_2 = G_8^5$.

Remark 4. *The associated abstract groups are uncovered with GAP4* [6] *by defining a* `g := FreeGroup()` *with $d-1$ generators, where d is the order of the group, feeding the entire Cayley table as d^2-1 generators into the group, and executing* `StructureDescription(g)`.

Two examples of normal series are

$$H_2 \lhd H_4^{10} \lhd H_8^{10,17,22} \lhd H_{16}^{2,10,17,18,20,21,22} \lhd G_{32}^{49} \tag{7}$$

$$H_2 \lhd H_4^{10} \lhd H_8^{2,10,18} \lhd H_{16}^{2,3,4,10,16,18,23} \lhd G_{32}^{49} \tag{8}$$

According to the first Sylow Theorem, each group $H_d^{\cdot\cdot}$ is member of at least one of such series.

3.2. **Factorizations.** The Floretion Group does not have representations as a direct product of subgroups, but as semi-direct products. One example for each of the various types of the sub-groups is:

$$G_{32}^{49} = H_{16}^{3,11,15,18,21,22,23} \wedge M_2^{26} \tag{9}$$

$$= H_{16}^{2,3,4,10,16,18,23} \wedge M_2^{15} \tag{10}$$

$$= H_8^{2,3,4} \wedge M_4^{15,25,26} \tag{11}$$

$$= H_8^{2,9,15} \wedge M_4^{18,20,26} \tag{12}$$

$$= H_8^{2,17,21} \wedge M_4^{19,29,32} \tag{13}$$

$$= H_8^{15,16,17} \wedge M_4^{18,21,22}. \tag{14}$$

The first of these equations means $G_{32}^{49} \leftrightarrow G_{16}^{11} \wedge C_2$. This is helpful because a 4-dimensional representation is induced from the tabulation of the eight 1-dimensional and two 2-dimensional representations of G_{16}^{11} [11, Group 16/6]. (The dimensionality of the induced rep. is the product of the dimensionality of the rep. of the subgroup times the index of the subgroup [9, §4]). In particular, besides the unit matrix for

g_1, GAP4 [6] proposes for the 5 generators g_{10}, $g_3 g_2^{-1} g_9$, $g_2^{-2} g_9$, $g_{11} g_3^{-1}$ and g_2^{-2}

$$p_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad p_{10} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \tag{15}$$

$$p_{22} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \quad p_{12} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \tag{16}$$

$$p_{30} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \quad p_5 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{17}$$

Definition 4. *(Set of 2×2 unimodular integer base matrices)* $\tau_I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $\tau_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\tau_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. $\tau_2 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

We deduce the other elements by reverse look-up in the multiplication table, using $\tau_1^2 = \tau_3^2 = \tau_I$, $\tau_2^2 = -\tau_I$, $\tau_1\tau_2 = \tau_3$, $\tau_1\tau_3 = \tau_2$, $\tau_2\tau_3 = \tau_1$, $\tau_3\tau_2 = -\tau_1$:

$$p_1 = \begin{pmatrix} \tau_I & 0 \\ 0 & \tau_I \end{pmatrix}; p_2 = \begin{pmatrix} 0 & \tau_2 \\ \tau_2 & 0 \end{pmatrix}; p_3 = \begin{pmatrix} 0 & -\tau_I \\ \tau_I & 0 \end{pmatrix}; p_4 = \begin{pmatrix} \tau_2 & 0 \\ 0 & -\tau_2 \end{pmatrix}; \tag{18}$$

$$p_{j+4} = -p_j, \quad j = 1, \dots, 4, \tag{19}$$

$$p_9 = \begin{pmatrix} -\tau_2 & 0 \\ 0 & -\tau_2 \end{pmatrix}; p_{10} = \begin{pmatrix} 0 & -\tau_1 \\ \tau_1 & 0 \end{pmatrix}; p_{11} = \begin{pmatrix} 0 & -\tau_3 \\ \tau_3 & 0 \end{pmatrix}; \tag{20}$$

$$p_{j+3} = -p_j, \quad j = 9, \dots, 11, \tag{21}$$

$$p_{15} = \begin{pmatrix} 0 & \tau_I \\ \tau_I & 0 \end{pmatrix}; p_{16} = \begin{pmatrix} -\tau_1 & 0 \\ 0 & -\tau_1 \end{pmatrix}; p_{17} = \begin{pmatrix} 0 & -\tau_1 \\ -\tau_1 & 0 \end{pmatrix}; \tag{22}$$

$$p_{18} = \begin{pmatrix} -\tau_3 & 0 \\ 0 & \tau_3 \end{pmatrix}; p_{19} = \begin{pmatrix} \tau_1 & 0 \\ 0 & -\tau_1 \end{pmatrix}; p_{20} = \begin{pmatrix} 0 & \tau_2 \\ -\tau_2 & 0 \end{pmatrix}; \tag{23}$$

$$p_{21} = \begin{pmatrix} -\tau_3 & 0 \\ 0 & -\tau_3 \end{pmatrix}; p_{22} = \begin{pmatrix} -\tau_I & 0 \\ 0 & \tau_I \end{pmatrix}; p_{23} = \begin{pmatrix} 0 & \tau_3 \\ \tau_3 & 0 \end{pmatrix}; \tag{24}$$

$$p_{j+9} = -p_j, \quad j = 15, \dots 23. \tag{25}$$

Remark 5. *The $|G_{32}^{49}| = 32$ 4×4 matrices of this matrix representation of G_{32}^{49} exhaust all possible matrices subject to the conditions: (i) They are block-diagonal or block-anti-diagonal containing only 2×2 submatrices of the form τ_j, $j = I, 1, 2, 3$. (ii) The two indices of the τ along the diagonal or anti-diagonal are the same. (Proof: there are 8 different signed $\pm\tau_j$ which can be placed in either the upper left or upper right corner for a total of 16, and keeping or switching a sign for the element in the opposite corner introduces two times as many elements.)*

Remark 6. *This matrix representation is "faithful" in the sense that multiplication of the matrix by -1 yields the representation of the element with a switched sign in its name.*

The GAP function `GeneratorsSmallest` computes a generating set of only 4 elements, g_2, g_3, g_9 and g_{10}. If these elements are used as generators, the Cayley graph of Fig. 2 is obtained: Each of the 32 elements is a node (shown by its index in the table). Nodes are connected by edges if one can obtain one from the other by multiplication with a generator; the color of the edge indicates which generator is in use. All edges are directed because none of these four generators it its own inverse.

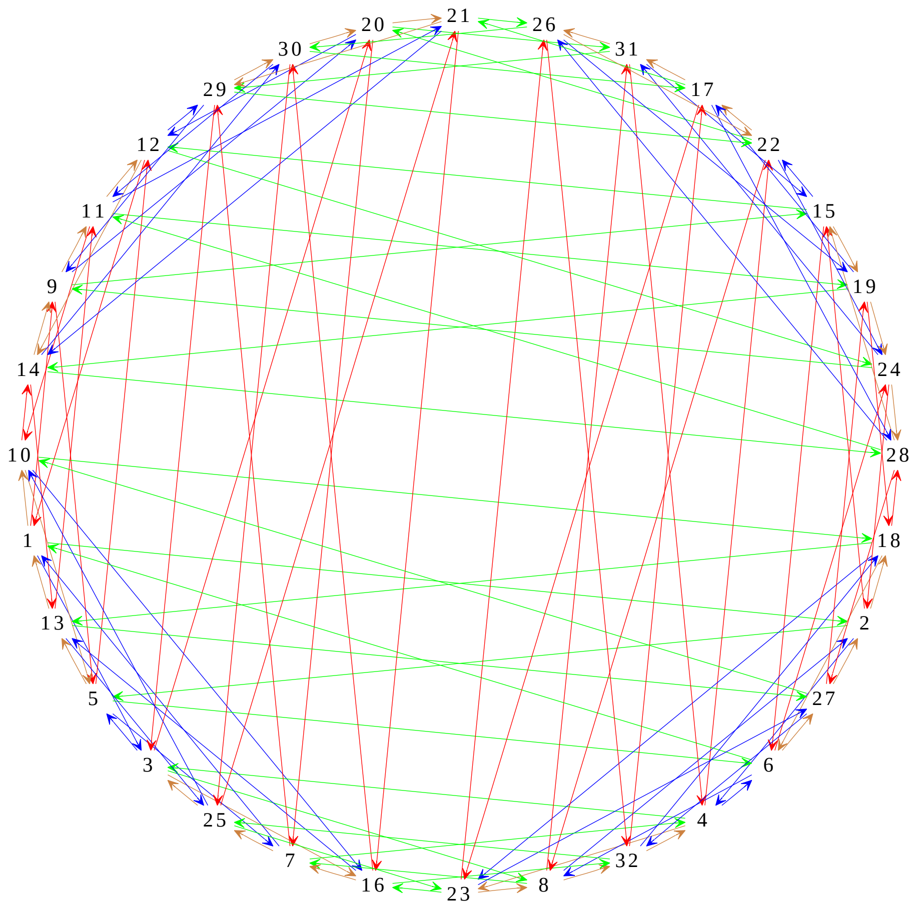

FIGURE 2. The Cayley graph from the four generators g_3 (blue), g_9 (red), g_2 (green), and g_{10} (brown).

APPENDIX A. THE GROUP $Q_8 \times Q_8$

The direct product $Q_8 \times Q_8 = G_{64}^{239}$ is connected to the Floretion group because the Factor Group $G_{64}^{239}/\{(g_1, g_1), (g_5, g_5)\}$ is isomorphous to G_{32}^{49}—as mentioned earlier by Pieper-Seier [4].

Remark 7. *The 64 elements of* G_{64}^{239} *are named* (g_i, g_j), $1 \le i, j, \le 8$, *using pairs of elements of Table 1.*

The non-Abelian G_{64}^{239} has 4 minimal generators, p-Rank 2. The character table can be obtained from the 156th group of order 64 in the Schaps compilation [10].

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