P-FINITE RECURRENCES FROM SQRT GENERATING FUNCTIONS

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Abstract. We derive the P-finite recurrences for some special sequences with arithmetic ordinary generating function containg roots of polynomials. The focus is on establishing the D-finite differential equations and their power series expansions.

1. Inverse Root

Let $g(x)$ be the (ordinary) generating function of a sequence which is an inverse r-th root of a polynomial $p(x)$ of degree o,

$$g(x) \equiv \sum_{n\ge 0} a_n x^n = \frac{1}{\sqrt[r]{p(x)}}, \quad r \neq 0. \tag{1}$$

$$p(x) \equiv \sum_{n=0}^{o} p_n x^n. \tag{2}$$

Then the chain rule of differentiation yields a first derivative of (1),

$$g'(x) = -\frac{1}{r}\frac{p'(x)}{p^{1+1/r}(x)} = -\frac{1}{r}\frac{p'(x)}{p(x)} g(x). \tag{3}$$

Obviously $g(x)$ is a D-finite function [9, 4]:

$$rp(x)g'(x) + p'(x)g(x) = 0. \tag{4}$$

The P-finite recurrence is derived by insertion of the two power series:

$$r\sum_{i=0}^{o} p_i x^i \sum_{j\ge 1} j a_j x^{j-1} + \sum_{i=1}^{o} i p_i x^{i-1} \sum_{j\ge 0} a_j x^j = 0. \tag{5}$$

Resummation with $k \equiv i+j-1$ yields

$$r\sum_{k=0}^{o-1}\sum_{j=1}^{k+1} p_{k+1-j} j a_j x^k + r\sum_{k\ge o}\sum_{j=k+1-o}^{k+1} p_{k+1-j} j a_j x^k$$
$$+\sum_{k=0}^{o-2}\sum_{j=0}^{k}(k+1-j)p_{k+1-j}a_j x^k + \sum_{k\ge o-1}\sum_{j=k+1-o}^{k}(k+1-j)p_{k+1-j}a_j x^k = 0. \tag{6}$$

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Comparison of coefficients for x^k, $k \geq o$, on both sides gives

$$r \sum_{j=k+1-o}^{k+1} p_{k+1-j} j a_j + \sum_{j=k+1-o}^{k} (k+1-j) p_{k+1-j} a_j = 0, \quad k \geq o. \tag{7}$$

Setting $n \equiv k+1$ gives

$$r \sum_{j=n-o}^{n} j p_{n-j} a_j + \sum_{j=n-o}^{n-1} (n-j) p_{n-j} a_j = 0, \quad n \geq o+1. \tag{8}$$

$$rnp_0 a_n + \sum_{j=n-o}^{n-1} [rj + n - j] p_{n-j} a_j = 0, \quad n \geq o+1. \tag{9}$$

Setting $j = n - l$ gives

$$rnp_0 a_n + \sum_{l=1}^{o} (rn - rl + l) p_l a_{n-l} = 0, \quad n \geq o+1. \tag{10}$$

As a natural extension of Noe's [6] recurrences we find:

Theorem 1. *The coefficients of the generating function (1) obey the P-finite recurrence*

$$\sum_{l=0}^{o} (rn - rl + l) p_l a_{n-l} = 0, \quad n \geq o+1. \tag{11}$$

Remark 1. *Note that for $r = 1$ this equation may be divided through a common factor n,*

$$\sum_{l=0}^{o} p_l a_{n-l} = 0, \quad n \geq o+1, \quad r = 1, \tag{12}$$

which is a recurrence with constant coefficients p_l. We recover the well known result: The sequences with a rational generating function $p(x)/q(x)$ have recurrences where the D-finite equation does not contain derivatives of $g(x)$, so the $a(n)$ obey C-finite recurrences.

Remark 2. *Our formulas for recurrences normalize the representation by using indices a_{n-l}, $l \geq 0$, because that is the most readable convention while implementing computer programs that derive sequences from lower-order terms [5, 7].*

Example 1. *Examples of (1), square roots $r = 2$, degree $o = 1, 2$*

$p(x)$	
$1-2x-3x^2$	*A002426*
$1-6x-3x^2$	*A122868*
$1-6x+x^2$	*A001850*
$1-6x+5x^2$	*A026375*
$1-4x-4x^2$	*A006139*
$1-4x$	*A000984*
$1-2x+5x^2$	*A098331*
$1-4x^2$	*A126869*

Example 2. *Examples of (1), square roots $r = 2$, degree $o = 3$*

$p(x)$	
$1 - 4x^2 - 4x^3$	*A115962*
$1 - 2x - 7x^2 + 8x^3$	*A098477*
$1 - 2x - 3x^2 + 4x^3$	*A026569*
$1 - 2x - 3x^2 - 4x^3$	*A191354*
$1 - 2x + x^2 - 4x^3$	*A098479*
$1 - 2x + x^2 - 8x^3$	*A098480*
$1 - 4x - 8x^2 - 4x^3$	*A137635*
$1 - 4x + 8x^3$	*A165431*
$1 - 4x + 4x^3$	*A157004*

Example 3. *Examples of (1), roots $r \neq 2$*

$p(x)$	r	
$1 - 4x$	$2/3$	*A002457*
$1 - 8x$	$2/3$	*A115902*
$1 - 36x$	$6/11$	*A004998*
$1 + 9x + 9x^3$	-3	*A298308*
$1 - 9x - 27x^3$	3	*A095776*

2. Generalized Inverse Root

The case with a non-trivial numerator polynomial $q(x)$ of degree $\bar{o}$ and denominator polynomial $v(x)$ with degree $\hat{o}$ generalizes the content of Chapter 1:

$$g(x) = \frac{q(x)}{v(x)\sqrt[r]{p(x)}}, \tag{13}$$

where expansion coefficients q_n and v_n are defined via

$$q(x) \equiv \sum_{n=0}^{\bar{o}} q_n x^n. \tag{14}$$

$$v(x) \equiv \sum_{n=0}^{\hat{o}} v_n x^n. \tag{15}$$

Multiply (13) by $v(x)$, omitting the argument x for brevity:

$$vg = qp^{-1/r} \tag{16}$$

The derivative of (13) with respect to x with the Chain Rule:

$$\begin{aligned} vg' + v'g &= q'p^{-1/r} - \frac{1}{r}qp'p^{-1-1/r} \\ &= q'[qp^{-1/r}/v]\frac{v}{q} - \frac{1}{r}p'v\frac{qp^{-1-1/r}}{v} \\ &= q'g\frac{v}{q} - \frac{1}{r}p'v\frac{qp^{-1/r}}{pv} \\ &= q'g\frac{v}{q} - \frac{1}{r}p'v\frac{1}{p}g. \end{aligned} \tag{17}$$

Multiply by rqp and obtain a first-order D-finite differential equation with polynomial coefficients:

$$rqpvg' + rqpv'g = rpq'vg - qp'vg. \tag{18}$$

$$rqpvg' + (rqpv' - rpq'v + qp'v)g = 0. \tag{19}$$

Define two auxiliary polynomials of degree $\deg R$ with coefficients R_n and of degree $\deg Q$ with coefficients Q_n:

$$R(x) \equiv rqpv = \sum_{n=0}^{\deg R} R_n x^n; \quad Q(x) \equiv rqpv' - rpq'v + qp'v = \sum_{n=0}^{\deg Q} Q_n x^n, \tag{20}$$

such that (19) reads

$$Rg' + Qg = 0. \tag{21}$$

Remark 3. (13) *is a convolution of the sequence with generating function q/v by the sequence with generating function* (1)*. Compatible with Stoll's remark* [10]*, the product $R = rqpv$ indicates that the product of the degrees of the recurrences of the convoluted sequences is a bound for the degree of the recurrence.*

$$\sum_{i=0}^{\deg R} R_i x^i \sum_{j\ge 1} j a_j x^{j-1} + \sum_{i=0}^{\deg Q} Q_i x^i \sum_{j\ge 0} a_j x^j = 0. \tag{22}$$

$$\sum_{k=0}^{\deg R-1} \sum_{j=1}^{k+1} R_{k+1-j} j a_j x^k + \sum_{k\ge \deg R} \sum_{j=k+1-\deg R}^{k+1} R_{k+1-j} j a_j x^k + \sum_{k=0}^{\deg Q-1} \sum_{j=0}^{k} Q_{k-j} a_j x^k + \sum_{k\ge \deg Q} \sum_{j=k-\deg Q}^{k} Q_{k-j} a_j x^k = 0. \tag{23}$$

Comparison of coefficients $[x^k]$ for sufficiently large k on both sides yields the P-finite recurrence

$$\sum_{j=k+1-\deg R}^{k+1} j R_{k+1-j} a_j + \sum_{j=k-\deg Q}^{k} Q_{k-j} a_j = 0, \quad k \ge \max(\deg R, \deg Q). \tag{24}$$

Flip the direction in both j-sums:

$$\sum_{j=0}^{\deg R} (k+1-j) R_j a_{k+1-j} + \sum_{j=0}^{\deg Q} Q_j a_{k-j} = 0, \quad k \ge \max(\deg R, \deg Q). \tag{25}$$

Substitute $k+1 = n$:

$$\sum_{j=0}^{\deg R} (n-j) R_j a_{n-j} + \sum_{j=0}^{\deg Q} Q_j a_{n-j-1} = 0, \quad n > \max(\deg R, \deg Q). \tag{26}$$

Replace $j \to j-1$ in the second term:

$$\sum_{j=0}^{\deg R} (n-j) R_j a_{n-j} + \sum_{j=1}^{\deg Q+1} Q_{j-1} a_{n-j} = 0, \quad n > \max(\deg R, \deg Q). \tag{27}$$

In the case of (20) the degree of Q is one less than the degree of R because it contains one more derivative. If we define coefficients Q_n or R_n to be zero if $n < 0$ or n larger than the degree, this may be condensed as:

Theorem 2. *The P-finite recurrence of a sequence with the generating function* (13) *is*

$$\sum_{j=0}^{\deg R} [(n-j)R_j + Q_{j-1}]a_{n-j} = 0, \quad n > \deg R, \tag{28}$$

where $R(x)$ and $Q(x)$ are the polynomials defined by the sums and derivatives (20) *of the three polynomials $p(x)$, $q(x)$ and $v(x)$.*

Remark 4. *To keep the recurrences simple, common polynomial factors of x in homogeneous differential equations like* (21) *should be eliminiated (for example by finding the greatest common divisor with the Euclidean algorithm) before defining R and Q.*

Example 4. *Examples of* (13) *with square roots $r = 2$:*

$q(x)$	$p(x)$	v	
$1-x$	$1-6x+x^2$	*1*	*A110170*
$1+x$	$1-6x+x^2$	*1*	*A241023*
$1-x$	$1-6x+5x^2$	*1*	*A085362*
$1-x$	$1-2x-3x^2$	*1*	*A025178*
$1+x$	$1-2x-3x^2$	*1*	*A025565*
$1+2x$	$1-4x^2$	*1*	*A063886*
$1+x$	$1+4x^2$	*1*	*A128057*
1	$1-4x$	$1-x^2$	*A106188*
1	$1+4x$	$1-4x$	*A091520*

We did not require that r is an integer or positive. So formats like

$$g(x) = q(x)\sqrt{p(x)}, \quad r = -2 \tag{29}$$

or

$$g(x) = \frac{q(x)}{p^{3/2}(x)}, \quad r = 2/3 \tag{30}$$

are also covered.

The format

$$g = \sqrt{q(x)/p(x)} \tag{31}$$

is reduced to the format (13) by multiplying numerator and denominator by $\sqrt{q(x)}$ such that the numerator is root-free. The shortcut is:

Theorem 3. *The generating function* (31) *obeys the differential equation*

$$Rg' + Qg = 0, \tag{32}$$

with polynomials $R \equiv 2qp$ and $Q = qp' - q'p$, such that (28) *applies.*

If an additive polynomial $w(x)$ appears on the right hand side like

$$g(x) = w(x) + \frac{q(x)}{v(x)\sqrt[r]{p(x)}}, \tag{33}$$

this modifies the coefficients a_n for n up to the degree of the polynom $w(x)$. It delays the validity of (28) to the point that all indices j of the coefficients a_j must be larger than the degree of $w(x)$, so n in Theorem 2 must be larger than the sum of $\tilde{o}$ and the polynomial degree of w.

3. Generalized Inverse Root II

3.1. **Rooted Denominator.** The case with a non-trivial numerator polynomial $q(x)$ and denominator polynomials $v(x)$ and $w(x)$ generalizes the content of Chapter 1:

$$g(x) = \frac{q(x)}{w(x) + v(x)\sqrt[r]{p(x)}}, \tag{34}$$

where expansion coefficients q_n, v_n and w_n are defined via

$$q(x) \equiv \sum_{n\geq 0} q_n x^n; \tag{35}$$

$$v(x) \equiv \sum_{n\geq 0} v_n x^n; \tag{36}$$

$$w(x) \equiv \sum_{n\geq 0} w_n x^n. \tag{37}$$

With the strategy of Section 2 one ends up with a differential equation which contains terms proportional to g^2, which (apparently) does not lead to recurrences with a finite number of terms.

Some progress can be made in the case of square roots, $r = 2$, multiplying numerator and denominator of the fraction by $w - v\sqrt{p}$:

$$g(x) = \frac{q(x)[w(x) - v(x)\sqrt{p(x)}]}{w^2(x) - v^2(x)p(x)}, \tag{38}$$

$$(w^2 - v^2p)g = wq - vq\sqrt{p}. \tag{39}$$

The first derivative is

$$\begin{aligned}(w^2 - v^2p)g' + (w^2 - v^2p)'g &= (wq)' - (vq)'\sqrt{p} - vqp'\frac{1}{2\sqrt{p}} \\ &= (wq)' - \left[(vq)' + vq\frac{p'}{2p}\right]\sqrt{p}.\end{aligned} \tag{40}$$

Multiply by $2p$ to eliminate all denominators,

$$2p(w^2 - v^2p)g' + 2p(w^2 - v^2p)'g = 2p(wq)' - [2(vq)'p + vqp']\sqrt{p}. \tag{41}$$

To discard the square root, multiply this equation by vq, multiply (39) by $2(vq)'p + vqp'$,

$$2pvq(w^2 - v^2p)g' + 2pvq(w^2 - v^2p)'g = 2pvq(wq)' - vq[2(vq)'p + vqp']\sqrt{p}; \tag{42}$$

$$[2(vq)'p + vqp'](w^2 - v^2p)g = [2(vq)'p + vqp']wq - vq[2(vq)'p + vqp']\sqrt{p}. \tag{43}$$

and subtract both equations

$$2pvq(v^2p - w^2)g' + 2pvq(v^2p - w^2)'g + [2(vq)'p + vqp'](w^2 - v^2p)g = -2pvq(wq)' + [2(vq)'p + vqp']wq. \tag{44}$$

This is a first order differential equation with polynomial coefficients $R(x)$, $Q(x)$ and $H(x)$,

$$R(x)g' + Q(x)g = H(x), \tag{45}$$

where

$$R(x) \equiv 2pvq(v^2p - w^2) \equiv \sum_{j=0}^{\deg R} R_j x^j; \tag{46}$$

(47)

$$Q(x) \equiv -4pvqww' + 2pq(pv^2+w^2)v' + vq(pv^2+w^2)p' - 2pv(pv^2-w^2)q' \equiv \sum_{j=0}^{\deg Q} Q_j x^j;$$

$$H(x) \equiv -2pvq^2w' + 2wq^2pv' + wq^2vp'. \tag{48}$$

Remark 5. *For $w = 0$ these reduce to $H = 0$, $R = 2p^2v^3q$, $Q = pv^2(2pqv' + vqp' - 2pvq')$. The differential equation can be divided by the common factor pv^2 of R and Q because H is zero, and (20)–(21) emerge as a special case.*

Remark 6. *Unlike (45), holonomic functions are defined to obey a differential equation where no term such as $H(x)$ exists, which is not g or a derivative of g. That format is obtained by differntiating (45) d/dx $1 + \deg H$ times, such that H disappears in the final higher-order differential equation. The philosophy in this paper is to keep the order of the differential equation as low as possible, even if that means that the recurrence steps in later due to the influence of the $H(x)$ on the low powers of x.*

The further reduction follows exactly the path of Section 2, paying attention to eliminate the early a-coefficients where the low, non-vanishing orders of $H(x)$ may interfere:

Theorem 4. *The P-finite recurrence of a sequence with the generating function (34) at $r = 2$ is*

$$\sum_{j=0}^{\deg R} [(n-j)R_j + Q_{j-1}]a_{n-j} = 0, \quad n - \deg R > \deg H, \tag{49}$$

where $R(x)$ and $Q(x)$ are the polynomials defined by the sums and derivatives (46)–(47) of the four polynomials $p(x)$, $q(x)$, $v(x)$ and $w(x)$.

Example 5. *Examples of (34):*

p	q	v	w	r	
$1-2x-3x^2$	*1*	$1+x$	$-x$	*2*	*A116394*

3.2. **Rooted Numerator.**

$$g(x) = \frac{w(x) + v(x)\sqrt[r]{p(x)}}{q(x)}; \tag{50}$$

$$qg = w + v\sqrt[r]{p}; \tag{51}$$

$$qg' + q'g = w' + v'\sqrt[r]{p} + \frac{1}{r}v\frac{p'}{p}\sqrt[r]{p} = w' + v'\sqrt[r]{p} + \frac{1}{r}\frac{p'}{p}(qg - w); \tag{52}$$

Multiply this equation by v, the penultimate equation by v' and subtract to eliminate $\sqrt[r]{p}$:

$$qvg' + q'vg = w'v + vv'\sqrt[r]{p} + \frac{1}{r}v\frac{p'}{p}(qg - w); \tag{53}$$

$$qv'g = wv' + vv'\sqrt[r]{p}; \tag{54}$$

$$qvg' + (q'v - qv')g = w'v - wv' + \frac{1}{r}v\frac{p'}{p}(qg - w); \tag{55}$$

$$rpqvg' + rp(q'v - qv')g = rp(w'v - wv') + vp'(qg - w); \tag{56}$$

This fits (45), this time with

$$R(x) \equiv rpqv = \sum_{n=0}^{\deg R} R_n x^n; \tag{57}$$

$$Q(x) \equiv rp(q'v - qv') - vp'q = \sum_{n=0}^{\deg Q} Q_n x^n; \tag{58}$$

$$H(x) \equiv rp(w'v - wv') - vp'w; \tag{59}$$

Theorem 5. *The P-finite recurrence of a sequence with the generating function (50) is given by (49) where $R(x)$ and $Q(x)$ are the polynomials defined by the sums and derivatives (57)–(58) of the four polynomials $p(x)$, $q(x)$, $v(x)$ and $w(x)$.*

This formula and Section 1 cover Callan's generating functions [2].

The generating functions of the form

$$g = \frac{u(x)}{w(x)} + \frac{q(x)}{v(x)\sqrt[r]{p(x)}} \tag{60}$$

with polynomials $p(x)$, $v(x)$, $u(x)$ and $w(x)$ are also covered by the form (50) because they can be rewritten as

$$g = \frac{u(x)v(x)p(x) + w(x)q(x)p^{1-1/r}(x)}{w(x)v(x)p(x)}. \tag{61}$$

This allows to find P-recurrences of sequence which are sums of C-finite sequences represented by $u(x)/w(x)$ and sequences represented by $q(x)/[v(x)\sqrt[r]{p(x)}]$, such as transiting from [3, A026375] to [3, A242586].

The form with a common square root in numerator and denominator is also in this class:

$$g = \frac{u(x)\sqrt{p(x)} + w(x)}{v(x)\sqrt{p(x)} + q(x)} = \frac{u(x)v(x)p(x) - q(x)w(x) + [w(x)v(x) - q(x)u(x)]\sqrt{p(x)}}{v^2(x)p(x) - q^2(x)}. \tag{62}$$

4. Generalized Hypergeometric Function

The Generalized Hypergeometric Functions ${}_pF_q(x)$ with a set of constant "numerators" $\{\alpha\}_p$ and "denominators" $\{\beta\}_q$ are another special case with simple P-finite recurrences [8]:

$$g(x) = x^t {}_pF_q(\{\alpha\}_p; \{\beta\}_q; x^r/c) = \sum_{n\geq 0} \frac{\prod_{i=1}^p (\alpha_i)_n}{\prod_{j=1}^q (\beta_j)_n} \frac{x^{t+rn}}{n!c^n}, \tag{63}$$

where $(\cdot)_{\cdot}$ are Pochhammer symbols and the nonzero coefficients of the power series are

$$a_{rn+t} = \frac{\prod_i (\alpha_i)_n}{\prod_j (\beta_j)_n} \frac{1}{n!c^n}. \tag{64}$$

The associate P-finite 2-term recurrence is

$$c(n+1) \prod_j (\beta_j + n) a_{rn+r+t} = \prod_i (\alpha_i + n) a_{rn+t}. \tag{65}$$

The generating function

$$(1-x)^\alpha = {}_1F_0(-\alpha;;x) \tag{66}$$

is a borderline case between (1) and (63). Also

$$\arcsin x = x {}_2F_1(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2) \tag{67}$$

and

$$\ln(1+x) = x {}_2F_1(1,1;2;-x) \tag{68}$$

fit in here [1, §0.7.2].

5. Nested Roots

Let

$$g(x) = \sqrt[r]{w(x) + \sqrt{p(x)}} \tag{69}$$

be a generating function with polynomials $w(x)$ and $p(x)$.

5.1. Reduction to $T(x)g'' + R(x)g' + Q(x) = 0$.

The first and second derivatives are

$$g' = \frac{1}{r}(w' + \frac{1}{2}\frac{p'}{\sqrt{p}})(w+\sqrt{p})^{1/r-1}. \tag{70}$$

$$g'' = \frac{1}{r}\left[(\frac{1}{r}-1)(w' + \frac{1}{2}\frac{p'}{\sqrt{p}})^2(w+\sqrt{p})^{\frac{1}{r}-2} + (w'' + \frac{1}{2}\frac{p''}{\sqrt{p}} - \frac{1}{4}\frac{p'^2}{p^{3/2}})(w+\sqrt{p})^{\frac{1}{r}-1}\right]. \tag{71}$$

With the ansatz

$$T(x)g'' + R(x)g' + Q(x)g = 0 \tag{72}$$

we *assume* that this generating function is D-finite with three polynomials $T(x)$, $R(x)$ and $Q(x)$. This requires

(73)
$$T(x)\frac{1}{r}\left[(\frac{1}{r}-1)(w'+\frac{1}{2}\frac{p'}{\sqrt{p}})^2(w+\sqrt{p})^{\frac{1}{r}-2}+(w''+\frac{1}{2}\frac{p''}{\sqrt{p}}-\frac{1}{4}\frac{p'^2}{p^{3/2}})(w+\sqrt{p})^{\frac{1}{r}-1}\right]$$
$$+R(x)\frac{1}{r}(w'+\frac{1}{2}\frac{p'}{\sqrt{p}})(w+\sqrt{p})^{1/r-1}+Q(x)(w+\sqrt{p})^{1/r}=0.$$

Multiplied by $r^2(w+\sqrt{p})^{2-\frac{1}{r}}$

$$T(x)\left[(1-r)(w'+\frac{1}{2}\frac{p'}{\sqrt{p}})^2+r(w''+\frac{1}{2}\frac{p''}{\sqrt{p}}-\frac{1}{4}\frac{p'^2}{p^{3/2}})(w+\sqrt{p})\right]$$
$$+R(x)r(w'+\frac{1}{2}\frac{p'}{\sqrt{p}})(w+\sqrt{p})+Q(x)r^2(w+\sqrt{p})^2=0. \quad (74)$$

(75)
$$T(x)\left[(1-r)(w'^2+w'\frac{p'}{\sqrt{p}}+\frac{1}{4}\frac{p'^2}{p})+r(w''w+\frac{1}{2}w\frac{p''}{\sqrt{p}}-\frac{1}{4}w\frac{p'^2}{p^{3/2}})+r(w''\sqrt{p}+\frac{1}{2}p''-\frac{1}{4}\frac{p'^2}{p})\right]$$
$$+R(x)r(w'w+\frac{1}{2}w\frac{p'}{\sqrt{p}}+w'\sqrt{p}+\frac{1}{2}p')+Q(x)r^2(w^2+2w\sqrt{p}+p)=0.$$

$$T(x)[w'^2+w'\frac{p'}{\sqrt{p}}+\frac{1}{4}\frac{p'^2}{p}-rw'^2-rw'\frac{p'}{\sqrt{p}}-\frac{1}{4}r\frac{p'^2}{p} \quad (76)$$
$$+rw''w+\frac{1}{2}rw\frac{p''}{\sqrt{p}}-\frac{1}{4}rw\frac{p'^2}{p^{3/2}}+rw''\sqrt{p}+\frac{1}{2}rp''-\frac{1}{4}r\frac{p'^2}{p}]$$
$$+R(x)(rw'w+\frac{1}{2}rw\frac{p'}{\sqrt{p}}+rw'\sqrt{p}+\frac{1}{2}rp')+Q(x)(r^2w^2+2r^2w\sqrt{p}+r^2p)=0.$$

$$T(x)[(1-r)w'^2+(1-r)w'\frac{p'}{\sqrt{p}}+(\frac{1}{4}-\frac{r}{2})\frac{p'^2}{p} \quad (77)$$
$$+rw''w+\frac{1}{2}rw\frac{p''}{\sqrt{p}}-\frac{1}{4}rw\frac{p'^2}{p^{3/2}}+rw''\sqrt{p}+\frac{1}{2}rp'']$$
$$+R(x)(rw'w+\frac{1}{2}rw\frac{p'}{\sqrt{p}}+rw'\sqrt{p}+\frac{1}{2}rp')+Q(x)(r^2w^2+2r^2w\sqrt{p}+r^2p)=0.$$

This has the structure

(78)
$$T(x)[\alpha_1(x)+\frac{1}{p^{3/2}}\alpha_4(x)]+R(x)[\alpha_2(x)+\frac{1}{p^{3/2}}\alpha_5(x)]+Q(x)[\alpha_3(x)+\frac{1}{p^{3/2}}\alpha_6(x)]=0,$$

where 6 coefficients are rational polynomials in x defined as

$$\alpha_1(x) \equiv (1-r)w'^2 + (\frac{1}{4}-\frac{r}{2})\frac{p'^2}{p} + rw''w + \frac{1}{2}rp''; \tag{79}$$

$$\alpha_2(x) \equiv rw'w + \frac{1}{2}rp'; \tag{80}$$

$$\alpha_3(x) \equiv r^2w^2 + r^2p; \tag{81}$$

$$\alpha_4(x) \equiv (1-r)w'p'p + \frac{1}{2}rwp''p - \frac{1}{4}rwp'^2 + rw''p^2; \tag{82}$$

$$\alpha_5(x) \equiv \frac{1}{2}rwp'p + rw'p^2; \tag{83}$$

$$\alpha_6(x) \equiv 2r^2wp^2. \tag{84}$$

Instead of solving (78) in general we demand that the components which depend on $1/p^{3/2}$ and do not depend on it are individually zero:

$$T(x)\alpha_1(x) + R(x)\alpha_2(x) + Q(x)\alpha_3(x) = 0; \tag{85}$$

$$T(x)\alpha_4(x) + R(x)\alpha_5(x) + Q(x)\alpha_6(x) = 0. \tag{86}$$

In the language of 3-dimensional vector algebra, the vector (T, R, Q) is orthogonal to the vector $(\alpha_1, \alpha_2, \alpha_3)$ as well as to the vector $(\alpha_4, \alpha_5, \alpha_6)$, so it is the cross product of the two α-vectors:

$$T = \alpha_2\alpha_6 - \alpha_3\alpha_5; \tag{87}$$

$$R = \alpha_3\alpha_4 - \alpha_1\alpha_6; \tag{88}$$

$$Q = \alpha_1\alpha_5 - \alpha_2\alpha_4. \tag{89}$$

Expansion of the differences and multiplying T, R and Q with a common factor $8/r$:

$$T = 4r^2(-w^2+p)p(-2w'p + wp'); \tag{90}$$

(91)

$$R = -2r(-4w^2w'p'p + 4rw^2w'p'p - 2rw^3p''p + rw^3p'^2 + 4rw^2w''p^2 - 4p^2w'p' + 4rp^2w'p' + 2rp^2wp'' + 5rwpp'^2 - 4rp^3w'' + 8wp^2w'^2 - 8rwp^2w'^2 - 2wpp'^2);$$

$$Q = -4w'^2wp'p + 8w'^3p^2 + 4w'^2rwp'p - 8w'^3rp^2 - p'^3w - 6p'^2pw' + 3p'^3rw + 8p'^2prw' + 4rw''w^2p'p + 4rp''w'p^2 - 4rw'w^2p''p + 2rw'w^2p'^2 - 4rp'w''p^2. \tag{92}$$

Working backwards through the logic shows that the ansatz (72) is indeed satisfied.

Theorem 6. *The coefficients of the generating function (69) obey the P-finite recurrence*

$$\sum_{j\ge 0} \left[(n-j)(n-j-1)T_j + (n-j)R_{j-1} + Q_{j-2}\right] a_{n-j} = 0, \tag{93}$$

where $T = \sum_{n\ge 0} T_n x^n$, $R = \sum_{n\ge 0} R_n x^n$ *and* $Q = \sum_{n\ge 0} Q_n x^n$ *are the polynomials* (90)–(92).

5.2. **Special cases.** T of (90) is zero if $w^2 = p$ or $wp' = 2w'p$, and (21) applies; the P-recurrence only involves first-degree polynomials. The case $w^2 = p$ is not interesting: then $g = \sqrt[r]{2w}$ has the format (1) and would be treated accordingly. The case $wp' = 2w'p$ means $2w'/w = p'/p$, therefore $2\ln w = \ln p + C$, therefore $\ln w^2 = \ln p + C$, therefore $w^2 = Cp$, and again (1) is the underlying format.

If w is a constant, $w' = w'' = 0$, the equations simplify to

(94) $$T = 4r^2 p' p w(p - w^2);$$

(95) $$R = -2rw(w^2 r p'^2 - 2w^2 r p'' p - 2pp'^2 + 2rp^2 p'' + 5rpp'^2);$$

(96) $$Q = (3r-1)wp'^3.$$

6. Exponential of Root

The class of generating functions

(97) $$g(x) = \exp[w(x) \pm \sqrt{p(x)}]$$

has similar regenerative properties as the nested roots:

(98) $$g' = \left(w' \pm \frac{1}{2}p'p^{-1/2}\right)\exp[w \pm \sqrt{p}];$$

(99) $$g'' = \left(w'' \pm \frac{1}{2}p''p^{-1/2} \mp \frac{1}{4}p'^2 p^{-3/2}\right)\exp[w \pm \sqrt{p}] + \left(w' \pm \frac{1}{2}p'p^{-1/2}\right)^2 \exp[w \pm \sqrt{p}].$$

The same procedure as in Section 5 unfolds:

(100) $$T[\left(w'' \pm \frac{1}{2}p''p^{-1/2} \mp \frac{1}{4}p'^2 p^{-3/2}\right)\exp[w \pm \sqrt{p}] + \left(w' \pm \frac{1}{2}p'p^{-1/2}\right)^2 \exp[w \pm \sqrt{p}]] + R\left(w' \pm \frac{1}{2}p'p^{-1/2}\right)\exp[w \pm \sqrt{p}] + Q\exp[w \pm \sqrt{p}] = 0.$$

(101) $$T[w'' \pm \frac{1}{2}p''p^{-1/2} \mp \frac{1}{4}p'^2 p^{-3/2} + \left(w' \pm \frac{1}{2}p'p^{-1/2}\right)^2] + R\left(w' \pm \frac{1}{2}p'p^{-1/2}\right) + Q = 0.$$

(102) $$T[w'' \pm \frac{1}{2}p''p^{-1/2} \mp \frac{1}{4}p'^2 p^{-3/2} + w'^2 \pm w'p'p^{-1/2} + \frac{1}{4}p'^2 \frac{1}{p}] + R\left(w' \pm \frac{1}{2}p'p^{-1/2}\right) + Q = 0.$$

(103) $$\alpha_1 = w'' + w'^2 + \frac{1}{4}\frac{p'^2}{p};$$

(104) $$\alpha_2 = w';$$

(105) $$\alpha_3 = 1;$$

(106) $$\alpha_4 = \pm\frac{1}{2}p''p \mp \frac{1}{4}p'^2 \pm w'p'p;$$

(107) $$\alpha_5 = \pm\frac{1}{2}p'p;$$

(108) $$\alpha_6 = 0.$$

Obtain the three polynomials (87)–(89) and multiply T, R and Q by the common factor 8 to keep all terms polynomial:

$$T(x) = \mp 4p'p = \sum_{n\geq 0} T_n x^n; \tag{109}$$

$$R(x) = \pm 2(2p''p - {p'}^2 + 4w'p'p) = \sum_{n\geq 0} R_n x^n; \tag{110}$$

$$Q(x) = \pm(4p'pw'' - 4p'p{w'}^2 + {p'}^3 - 4w'p''p + 2w'{p'}^2) = \sum_{n\geq 0} Q_n x^n. \tag{111}$$

Theorem 7. *The coefficients of the generating function (97) obey the P-finite recurrence*

$$\sum_{j\geq 0} [(n-j)(n-j-1)T_j + (n-j)R_{j-1} + Q_{j-2}]\, a_{n-j} = 0, \tag{112}$$

where $T(x)$, $R(x)$ and $Q(x)$ are the polynomials (109)–(111).

Example 6. *The generating function* $\exp[1-x-\sqrt{1-4x+x^2}]$ *has $w = 1-x$, $p = 1-4x+x^2$, therefore $T = -16+72x-48x^2+8x^3$, $R = -8+144x-96x^2+16x^3$, $Q = 72-24x$. Because the right hand side of (72) is zero, we can dividie T, R, Q through a common factor 8: $T = -2+9x-6x^2+x^3$, $R = -1+18x-12x^2+2x^3$, $Q = 9-3x$. The long writeup of Theorem 7 is:*

$$\begin{gathered} n(n-1)(-2)a_n + [(n-1)(n-2)\cdot 9 + (n-1)(-1)]a_{n-1} + [(n-2)(n-3)\cdot(-6) + (n-2)\cdot 18 + 9]a_{n-2} \\ + [(n-3)(n-4) + (n-3)\cdot(-12) + (-3)]a_{n-3} + [(n-4)(n-5)\cdot 0 + (n-4)\cdot 2 + 0]a_{n-4} = 0. \end{gathered} \tag{113}$$

$$\begin{aligned} &-2n(n-1)a_n + (n-1)(9n-19)a_{n-1} + (-6n^2+48n-63)a_{n-2} \\ &\qquad + (n^2-19n+45)a_{n-3} + 2(n-4)a_{n-4} = 0. \end{aligned} \tag{114}$$

If we are concerned about the exponential *generating function* [3, A078634]*, the substitution $a_m \to a_m/m!$ is required:*

$$\begin{aligned} &-2n(n-1)\frac{a_n}{n!} + (n-1)(9n-19)\frac{a_{n-1}}{(n-1)!} + (-6n^2+48n-63)\frac{a_{n-2}}{(n-2)!} \\ &\qquad + (n^2-19n+45)\frac{a_{n-3}}{(n-3)!} + 2(n-4)\frac{a_{n-4}}{(n-4)!} = 0. \end{aligned} \tag{115}$$

$$\begin{aligned} &-2\frac{a_n}{(n-2)!} + (9n-19)\frac{a_{n-1}}{(n-2)!} + (-6n^2+48n-63)\frac{a_{n-2}}{(n-2)!} \\ &\qquad + (n^2-19n+45)(n-2)\frac{a_{n-3}}{(n-2)!} + 2(n-4)(n-3)(n-2)\frac{a_{n-4}}{(n-2)!} = 0. \end{aligned} \tag{116}$$

$$\begin{aligned} &-2a_n + (9n-19)a_{n-1} + (-6n^2+48n-63)a_{n-2} \\ &\qquad + (n^2-19n+45)(n-2)a_{n-3} + 2(n-4)(n-3)(n-2)a_{n-4} = 0. \end{aligned} \tag{117}$$

This is valid for $n \geq 2$, whereas the alternative (simpler)

$$\begin{aligned} &a_n + (-4n+9)a_{n-1} + (n^2-16n+26)a_{n-2} \\ &\qquad + (3n^2-23n+39)a_{n-3} + 2(n-4)(n-3)a_{n-4} = 0 \end{aligned} \tag{118}$$

is valid for $n \geq 3$. *This effect is induced by the* $\alpha_6 = 0$ *in* (108). *It means that one could rewrite* (78) *by moving the polynomial* $Q(x)\alpha_3(x)$ *to the right hand side, causing a "delayed" start of the recurrence.*

7. Exponential Generating Function

7.1. **Arithmetic.** If (13) is an *Exponential Generating Function*,

$$e(x) = \frac{q(x)}{v(x)\sqrt[r]{p(x)}} = \sum_{n\geq 0} a_n \frac{x^n}{n!}, \tag{119}$$

the first steps of Chapter 2 remain valid; once the Taylor series are inserted, a_j is to be replaced by $a_j/j!$:

$$\sum_{j=0}^{\deg R} \frac{1}{(n-j)!}[(n-j)R_j + Q_{j-1}]a_{n-j} = 0, \quad n > \deg R. \tag{120}$$

Multiply the equation by $(n-1)!$ to obtain polynomial coefficients:

Theorem 8. *The P-finite recurrence of a sequence with the Exponential Generating Function* (119) *is*

$$\sum_{j\geq 0} \frac{(n-1)!}{(n-j)!}[(n-j)R_j + Q_{j-1}]a_{n-j} = 0, \quad n > \deg R, \tag{121}$$

where $R(x)$ *and* $Q(x)$ *are the polynomials defined by the sums and derivatives* (20) *of the three polynomials* $p(x)$, $q(x)$ *and* $v(x)$.

Example 7. *Examples of* (119),

$p(x)$	$q(x)$	$v(x)$	r	
$1-4x+x^2$	1	1	2	A285199
$1-8x+x^2$	1	1	2	A006438
$1+2x+4x^2$	1	1	2	A182827
$1-2x-2x^2$	1	1	2	A098460
$1-2x-3x^2$	1	1	2	A098461
$1-10x$	1	1	10	A144773

7.2. **Exponential times Arithmetic.** If the exponential generating function is of the kind

$$e(x) = \exp[q(x)/v(x)]\frac{1}{\sqrt[r]{p(x)}} = \sum_{n\geq 0} a_n \frac{x^n}{n!}, \tag{122}$$

with polynomials $p(x)$, $q(x)$ and $v(x)$, then

$$e' = \left(\frac{q'}{v} - \frac{qv'}{v^2}\right)\exp(q/v)\frac{1}{\sqrt[r]{p}} - \frac{1}{r}\exp(q/v)\frac{p'}{p^{1+1/r}} = (\frac{q'}{v} - \frac{qv'}{v^2})e - \frac{1}{r}\frac{p'}{p}e. \tag{123}$$

Multiplication with rpv^2 yields the first order differential equation

$$rpv^2e' = [rp(q'v - qv') - v^2p']e; \tag{124}$$

$$Re' + Qe = 0, \quad R(x) \equiv rpv^2, \quad Q(x) \equiv v^2p' - rp(q'v - qv'). \tag{125}$$

Note that the degree of Q now equals the degree of R: (121) applies and includes all j until both sets of coefficients in R and Q are exhausted.

Example 8. *Examples of* (122),

$p(x)$	$q(x)$	$v(x)$	r	
$1-2x$	x	*1*	-2	*A055142*
$1-4x$	x	*1*	2	*A052143*
$1-8x$	$8x$	*1*	8	*A094935*
$1-7x$	$7x$	*1*	7	*A094911*

8. Summary

We validated a larger number of P-recurrences of sequences which involve generating function with roots.

Appendix A. Inhomogeneous P-finite

If the sequence a obeys a P-finite recurrence with a polynomial $I(n)$,

$$\sum_{j\ge 0} P_j(n)a_{n-j} + I(n) = 0, \tag{126}$$

it can be rewritten as a homogeneous P-finite recurrence by shifting the index by 1:

$$\sum_{j\ge 0} P_j(n-1)a_{n-j-1} + I(n-1) = 0, \tag{127}$$

multiplying (126) by $I(n-1)$ and multiplying (127) by $I(n)$ and subtracting both equations. This results in a recurrence which is one longer than (126) and has polynomial coefficients of degrees which are the sum of the degrees in (126) and the degree of $I(n)$.

Appendix B. Reduction Of the Number of Terms

In [3, A122877] the generating function

$$g = \frac{1-2x-3x^2-(1-x)\sqrt{1-2x-7x^2}}{8x^3} \tag{128}$$

matches (50) with parameters $q = 8x^3$, $w = 1-2x-3x^2$, $v = -(1-x)$, $p = 1-2x-7x^2$ and $r = 2$, such that the differential equation (45) is to be solved with $R = -16x^3(1-x)(1-2x-7x^2)$, $Q = -16x^2(3-7x-11x^2+7x^3)$ and $H = -64x^3$ defined in (57)–(59). The largest common factor $-16x^2$ of the R, Q and H can be dropped in the differential equation:

$$x(1-x)(1-2x-7x^2)g' + (3-7x-11x^2+7x^3)g = 4x. \tag{129}$$

Only the indices $j = 1$–4 contribute to the recurrence (49), so the generating function supports a 4-term recurrence:

$$(n+3)a_n - (3n+4)a_{n-1} - (5n+1)a_{n-2} + 7(n-2)a_{n-3} = 0 \tag{130}$$

with first degree polynomials. Differentiating of (129)

$$Rg' + Qg = H \tag{131}$$

yields a second order differential equation

$$Rg'' + (R'+Q)g' + Q'g = H', \tag{132}$$

here

$$x(1-x)(1-2x-7x^2)g'' + (1-x)(4-9x-35x^2)g' + (-7-22x+21x^2)g = 4. \tag{133}$$

The number of the terms in the recurrence derived from the first-order differential equation is based on:

- The factor R contributes powers $x^0 \ldots x^{\deg R}$; g' represents $\sum n a_n x^{n-1}$, so the product has powers x^{n-1} up to $x^{n-1+\deg R}$
- The factor Q contributes powers $x^0 \ldots x^{\deg Q}$; g represents $\sum a_n x^n$, so the product has powers x^n up to $x^{n+\deg Q}$.

The range of powers is x^{n-1} up to the larger of $x^{n-1+\deg R}$ or $x^{n+\deg Q}$, and the spread of exponents determines the number of coupled a-coefficients. The equivalent analysis of the second-order differential equation (using $g'' \equiv \sum n(n-1)a_n x^{n-2}$) shows that the exponents have been decremented by one, but the spread of exponents remains the same. [This preservation remains valid, even if some lower coefficients R_n vanish, like in our example where $R_0 = 0$.] The numbers of terms in the P-recurrences derived from (45) and (132) are the same.

The *penalty* in (132), induced by $g'' \sim \sum n(n-1)a_n$, is that the polynomials in the P-recurrences are of degree 2, not 1. *However*, if $\sum_n R_n = 0$ [equivalent: a factor $1-x$ in the factorization of $R(x)$], the contribution of the n^2 terms in the recurrence derived from (132) vanishes. In that circumstance the P-recurrence from (132) also has coefficients which are polynomials of *first* degree. In the synoptical view on both recurrences of the same number of terms and the same polynomial degrees, one may multiply each recurrence with the polynomial in front of a_n of the other recurrence, subtract both, to obtain a recurrence with one term less and with polynomial coefficients with a degree which is the sum of the individual degrees.

In the example considered here, the requirement on $R(x)$ is fulfilled, and besides (130) there is a 3-term recurrence

$$-(n+3)(n-1)a_n + n(2n+1)a_{n-1} + 7n(n-1)a_{n-2} = 0 \tag{134}$$

with quadratic polynomials.

Remark 7. *The derivative of D-finite differential equations with polynomial coefficients yields differential equations of higher order, equivalent to P-recurrences with polynomials of higher degrees, and potentially of a lower number of terms. We take the stand that recurrences derived from differential equations of lower order are preferable, even if the number of terms in the P-recurrences (the length of the recurrences) is larger, because the step from the P-recurrences to the D-equation plus differentiation is straight forward, whereas the opposite direction (one integration of the D-equation) may be difficult and introduces further constants.*

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