Energy Cascade and Scaling in Supersonic Turbulence

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1 Introduction

Turbulence is ubiquitous in molecular clouds (Elmegreen & Scalo 2004, Hennebelle & Falgarone 2012) and is believed to play a key role in regulating star formation (Mac Low & Klessen 2004, McKee & Ostriker 2007, Krumholz 2014). For several decades supersonic turbulence was used essentially as a *wild card* in star formation studies due to the lack of a simple conceptual theory that would capture the key physics and predict scaling of turbulent fluctuations. However, the situation is quickly evolving now due to progress in theory (Aluie 2013), new exact scaling relations derived from the Navier-Stokes (N-S) equations (e.g. Galtier & Banerjee 2011) and large-scale numerical experiments (Kritsuk, et al. 2013a,b; Federrath 2013).

Here we present a summary of results of a study confronting the new theory with numerical experiments and observational measurements (see Kritsuk, et al. 2013a,b for more detail).

2 New exact relation

We start with a brief outline of the derivation, highlighting the key conceptual elements, and then continue by evaluating individual terms in the new relation using data from a Mach 6 simulation of isothermal turbulence carried out with the PPM of Colella & Woodward (1984).

Consider the N-S equations for an isothermal compressible fluid

$$\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0,$$

$$\partial_t(\rho\boldsymbol{u}) + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}\boldsymbol{u}) + \boldsymbol{\nabla}p = \eta\Delta\boldsymbol{u} + \frac{\eta}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) + \boldsymbol{f},$$

where $p = c_s^2\rho$ is the pressure, $\eta > 0$ viscosity, and $\boldsymbol{f}(\boldsymbol{x},t)$ is a large-scale random force. Total energy density is one of ideal integral invariants of the system: $E \equiv \left\langle \rho\boldsymbol{u}^2/2 + \rho e\right\rangle$, where $e = c_s^2\ln(\rho/\rho_0)$ is the specific compressive "potential" energy.

The following energy balance equation describes competition between large scale energy injection and small-scale dissipation

$$\partial_t E = \langle\epsilon\rangle - \eta\left\langle\omega^2 + 4d^2/3\right\rangle,$$

were $\epsilon = \boldsymbol{u}\cdot\boldsymbol{f}$ is the local energy injection rate, $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$ vorticity, $d = \boldsymbol{\nabla}\cdot\boldsymbol{u}$ dilatation, and $\langle\ldots\rangle$ an ensemble average. In a statistical steady state in the limit of small viscosity η ($Re\to\infty$), the pumping and dissipation rates are finite and compensate each other exactly.

The following scaling relation (which additionally assumes isotropy)

$$Q(r) + F_{\|}(r) = -\tfrac{4}{3}\varepsilon r,$$

describes how the energy propagates in a fluid from large to small scales through the inertial range. Here the longitudinal energy flux $F_{\|}$, source Q and pumping rate ε are defined as

$$F_{\|}(r) = \left\langle[\delta(\rho\boldsymbol{u})\cdot\delta\boldsymbol{u} + 2\delta\rho\delta e]\,\delta u_{\|} + \tilde{\delta}e\delta(\rho u_{\|})\right\rangle,$$

$$Q(r) \equiv \frac{1}{r^2}\int_0^r S(r)r^2\mathrm{d}r,$$

$$S(r) = \left\langle\left[\delta(d\rho\boldsymbol{u}) - \tilde{\delta}d\delta(\rho\boldsymbol{u})\right]\cdot\delta\boldsymbol{u} + 2\left[\delta(d\rho) - \tilde{\delta}d\delta\rho\right]\delta e + \delta d\delta p - 2dp\right\rangle,$$

$$\varepsilon = \langle\rho\boldsymbol{u}'\cdot\boldsymbol{a} + \rho'\boldsymbol{u}'\cdot\boldsymbol{a}\rangle/2$$

(Galtier & Banerjee 2011; Kritsuk, et al. 2013a).

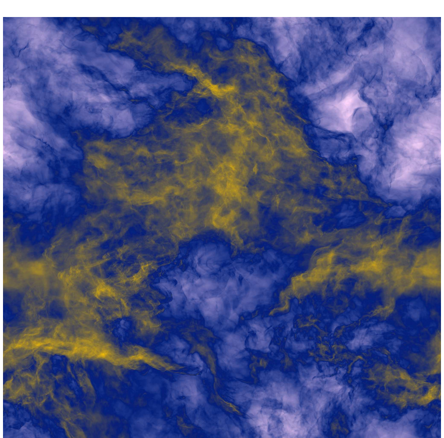

FIGURE 1: Projected gas density in a Mach 6 turbulence simulation at 2048^3.

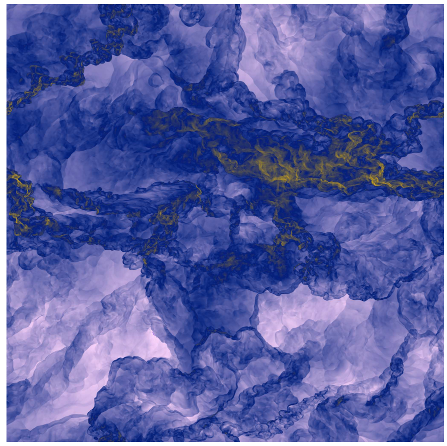

FIGURE 2: A density slice from the same simulation as Fig. 1. Most of the sharp structures seen are shock waves. Note how shocks are clustered around density peaks.

We used high resolution data from Kritsuk, et al. (2007) to verify the relation and access relative importance of different terms.

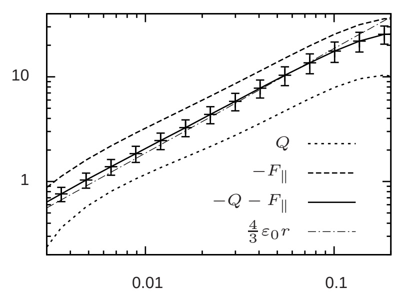

FIGURE 3: The new relation holds reasonably well and describes a direct energy cascade with an effective sink due to compressibility. Both flux $F_{\|}(r)$ and source $Q(r)$ scale approximately linearly with r and have opposite signs, $|F_{\|}|/Q \approx 3.2$. Origin of linear scaling of the source still remains unexplained.

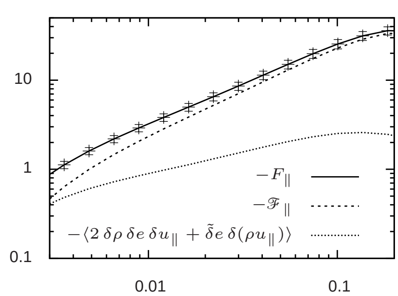

FIGURE 4: The flux $F_{\|}(r)$ can be reasonably well represented by the first (dominant) term $\mathscr{F}_{\|}(r)$. Contributions from terms that contain increments of compressive energy δe can be safely ignored in high Mach number turbulence.

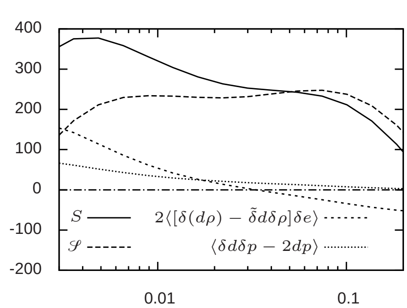

FIGURE 5: The source $S(r)$ can be well approximated by the first term $\mathscr{S}(r)$. The remaining terms dependent on the increment of compressive energy δe or pressure p are subdominant in high Mach number turbulence.

3 Scaling in supersonic turbulence

As the comparative analysis of relative contributions of different terms in the full exact relation shows, at high Mach numbers, the scaling relation can be substantially simplified. Let's define

$$\mathscr{F}_{\|}(r) = \left\langle[\delta(\rho\boldsymbol{u})\cdot\delta\boldsymbol{u}]\,\delta u_{\|}\right\rangle,$$

$$\mathscr{S}(r) = \left\langle\left[\delta(d\rho\boldsymbol{u}) - \tilde{\delta}d\delta(\rho\boldsymbol{u})\right]\cdot\delta\boldsymbol{u}\right\rangle,$$

$$\varepsilon(r) \approx \langle\rho\boldsymbol{u}\cdot\boldsymbol{a}\rangle = \varepsilon_0.$$

Then, ignoring all subdominant terms representing fluctuations of pressure p and compressive energy e, we obtain

$$\mathscr{Q}(r) + \left\langle[\delta(\rho\boldsymbol{u})\cdot\delta\boldsymbol{u}]\delta u_{\|}\right\rangle = -\frac{4}{3}\mathscr{C}\varepsilon_0 r,$$

where an order unity constant $\mathscr{C} \approx 0.87$ at Mach 6.

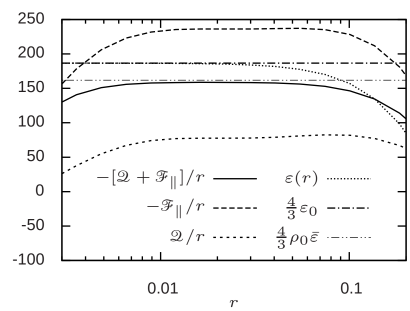

FIGURE 6: The dominant constituents in the energy flux and source terms demonstrate linear scaling over at least one decade in separation r. Note that the ordinate scale is linear.

Since $\mathscr{Q}(r) \propto r$, it can be included in $\epsilon_{\rm eff}$, leading to a compact form

$$\left\langle[\delta(\rho\boldsymbol{u})\cdot\delta\boldsymbol{u}]\delta u_{\|}\right\rangle = -\tfrac{4}{3}\varepsilon_{\rm eff}r,$$

which should be compared with a primitive version of the 4/5 law of Kolmogorov (1941) for incompressible turbulence

$$\left\langle(\delta\boldsymbol{u})^2\delta u_{\|}\right\rangle = -\frac{4}{3}\bar{\varepsilon}r,$$

where $\bar{\varepsilon} = \varepsilon_0/\rho_0$. This new scaling relation has a remarkable and unique property to asymptotically converge to the 4/5 law at Mach numbers $M \ll 1$ and also capture correct asymptotic behavior at $M \gg 1$. Moreover, it allows one to derive observed Larson's (1981) relations from first principles, supporting their interpretation in terms of supersonic turbulence (Kritsuk, et al. 2013b).

4 Conclusions

New Kolmogorov-like scaling relation for compressible isothermal turbulence is verified numerically at Mach 6. Our results support a direct energy cascade picture in three-dimensions developed earlier based on dimensional arguments (Kritsuk, et al. 2007). Both incompressible and highly supersonic limits permit major simplifications in the analytical treatment, while transonic regime is the most complex. Linear inertial range scaling of the compressible kinetic energy flux with separation $\left\langle[\delta(\rho\boldsymbol{u})\cdot\delta\boldsymbol{u}]\delta u_{\|}\right\rangle \propto r$ appears to be universal and is expected to hold at arbitrary Mach numbers.

Equipartition of energy between dilatational and solenoidal modes is expected in fully developed turbulence (Kraichnan 1955). In simulations with strongly nonequilibrium pumping (e.g. purely compressive), the universal scaling should still be seen at small separations far away from the energy injection scale, but yet above the dissipation scale, if the grid resolution is sufficiently high (cf. Federrath 2013).

5 References

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