Outflows & Jets: Theory & Observations

Lecture winter term 2008/2009

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## Outflows & Jets: Theory & Observations

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Outflows & Jets: Theory & Observations

Brief introduction to MHD

MHD concept: ionized, neutral, single fluid: average quantities:
\[ \vec{j} \equiv q_e \vec{v}_e \rho_e + q_i \vec{v}_i \rho_i \]

Ideal MHD: “frozen-in” field lines:

\[ \rightarrow \text{mass flux couples to magnetic flux} \]
\[ \rightarrow \text{matter moves “along” the field lines} \]

MHD Lorentz force:
\[ \vec{F}_L \sim \vec{j} \times \vec{B} \]

MHD equations (can only be solved numerically):

\[ \partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0 \]
\[ \rho (\partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v}) + \nabla P + \rho \nabla \Phi - \vec{j} \times \vec{B} = 0 \]
\[ \rho (\partial_t e + (\vec{v} \cdot \nabla) e) + P (\nabla \cdot \vec{v}) - \eta_D |\vec{j}|^2 / c^2 = 0 \]
\[ \partial_t \vec{B} = \nabla \times (\vec{v} \times \vec{B} - \eta_D \vec{j} / c) \]
\[ \nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{B} = 4\pi \vec{j} / c \]

(note non-ideal MHD resistive term of magnetic diffusivity)
Brief introduction to MHD

MHD concept: ionized, neutral, single fluid: average quantities:

Ideal MHD: “frozen-in” field lines

**MHD Lorentz force:**
\[ \vec{F}_L \sim \vec{j} \times \vec{B} \]

**Axisymmetric jets:**
-> poloidal, toroidal field components: \( B = B_p + B_\phi \)
-> magnetic flux surfaces:
\[ \Psi(R, Z) \sim \int \vec{B}_P \cdot d\vec{A} \]

**Lorentz force components 1:**
projected on \( \Psi \):
\[ \vec{F}_L \equiv \vec{F}_{L,\parallel} + \vec{F}_{L,\perp} \]

-> (de/) accelerating:
\[ \vec{F}_{L,\parallel} \equiv \vec{j}_\perp \times \vec{B}_\phi \]
-> (de-) collimating:
\[ \vec{F}_{L,\perp} \equiv \vec{j}_\parallel \times \vec{B} \]
Outflows & Jets: Theory & Observations

MHD theory

Three approaches to describe an ionized gas

- 1) test particles
- 2) plasma physics (two fluid components)
- 3) MHD (one fluid approach)

see e.g.: http://www.plasma-universe.com/index.php/Plasma-Universe.com
1) **ions & electrons**

spiral along magnetic field

Lorentz force: \( \vec{F} = q \vec{v} \times \vec{B} \)

gyro radius / **Larmor radius**: 
\[ r_L \equiv \frac{v_{\text{tan}}}{\Omega_L} \]

gyrofrequency/ cyclotron frequency: 
\[ \Omega_L = e B / m_{e,i} \]

\( r_L \ll L \Rightarrow \) gas moves on “straight” trajectories if 
\( L \gg r_L \Rightarrow \) gas is “magnetized” if characteristic length scale

**Magnetization parameter** (e, i): 
\[ \delta_{e,i} = \frac{r_L}{L} \]

magnetized plasma if 
\[ \delta_i = \frac{r_{L,i}}{L} \]
Outflows & Jets: Theory & Observations

MHD theory

1) ions & electrons

- examples:
  AGN jet (IC 4296): \( B \approx 10 \mu G, v \approx c_s \approx 0.005c, \rightarrow r_L = 0.03 \text{ cm} \)
  ISM, protons: \( B \approx 3 \mu G, 10^{17} \text{ eV}, \rightarrow r_L = 30 \text{ pc} \)
  Galactic field, protons: \( B \approx \mu G, 10^{19} \text{ eV}, \rightarrow r_L = 3 \text{ kpc} \)

- UHE Cosmic Rays \( \sim 10^{21} \text{ eV}, \) origin unknown;
  particles escape if \( L \ll r_L \)
  \( \rightarrow \) estimate of maximum energies in generation process

- Radiation from gyrating articles: cyclotron & synchrotron emission, Bremsstrahlung
  hot AGN jet plasma \( \rightarrow \) relativistic motion of particles \( \gamma < 1000 \)
  \( \rightarrow \) compare to relativistic bulk motion of jet \( \Gamma < 10 \)

See Larmor radius calculator @ http://pps.coe.kumamoto-u.ac.jp/streaming/PulsedPower/formulary/cal-lr.html
2) **plasma physics:** \( \rightarrow \) many particles \( \rightarrow \) statistical theory \( \rightarrow \) collective forces

see [http://farside.ph.utexas.edu/teaching/plasma/lectures/lectures.html](http://farside.ph.utexas.edu/teaching/plasma/lectures/lectures.html)

- **Quasi-neutrality:** number densities \( n_i \sim n_e \sim n \)

- **Plasma kinetic temperature** (in energy units): \( k T_{e,i} \equiv \frac{1}{3} m_{e,i} \langle v^2 \rangle \)

  thermal speed for \( T_i = T_e = T : v_{th,e,i} \equiv \sqrt{2 k T / m_{e,i}} \)

- **Plasma frequency:** collective dynamic behaviour: charge separation: \( \sigma = e n \, \delta x = -\epsilon_0 E_x \)

  \( \rightarrow \) electrostatic oscillation in electric field: \( m \frac{d^2}{dt^2} \delta x = e E_x = -m (\omega_p)^2 \, \delta x \)

  \( \rightarrow \) electron plasma frequency:
  \[ (\omega_{p,e})^2 = 4 \pi n_e e^2 / m_e \quad , \quad \omega_{p,e} [s^{-1}] = 5.64 \times 10^4 \, n_e^{1/2} [cm^{-3}] \]

  \( \rightarrow \) most fundamental time-scale in plasma physics

  \( \rightarrow \) observable only if 1) oscillation period \( \ll \) life time of system:

  2) external forcing slower than \( \omega_p \)

  \( \rightarrow \) if \( L < v_{th} / \omega_p \rightarrow \) plasma behaviour not detected, particles escape

  \( \rightarrow \) critical distance: **Debye length** \( \lambda_D \equiv (kT / m)^{1/2} \omega_p^{-1} \ll L \) for a plasma
Outflows & Jets: Theory & Observations

MHD theory

2) plasma physics

-> Debye shielding: calculate average Coulomb force by charged particles:

-> Coulomb potential of test charge Q

  -> without plasma: \( \Phi = \frac{Q}{r} \)

  -> within plasma: polarization: charge density \( \sigma = q (n' - n) \)

    (undisturbed and disturbed density of charges \( n \) and \( n' \))

-> Poisson equation: \( \Delta \Phi = -4 \pi \sigma - 4\pi Q \delta(r) \)

-> thermodynamic equilibrium:

    Boltzmann distribution of charged particles: \( n' = n \exp\left(-\frac{q \Phi}{kT}\right) \)

    -> Boltzmann potential \( \Phi \) should be local potential (not averaged)

    \( \langle \exp\left(-\frac{q \Phi}{kT}\right) \rangle \neq \exp\left(-q \langle \Phi \rangle/kT\right) \)

    -> Taylor expansion (far from charge Q): \( \langle \exp\left(-\frac{q \Phi}{kT}\right) \rangle = 1 - \langle q \Phi/kT \rangle \)

    and \( \sigma = -n q^2 \Phi/kT \)
2) **plasma physics**

- solution of Poisson equation

\[ \frac{d^2}{dr^2} (r \Phi) = \frac{1}{r_D^2} (r \Phi) \]

with B.C. \( \Phi \rightarrow \frac{Q}{r} \) for \( r \rightarrow 0 \)

\( \Phi \rightarrow 0 \) for \( r \rightarrow \) infinity

gives \( \Phi = \left( \frac{Q}{r} \right) \exp \left( -\frac{r}{\lambda_D} \right) \)

Debye length:

\[ \lambda_D = \left( \frac{kT}{4\pi n q^2} \right)^{1/2} \]

\[ \lambda_D = 743 T^{1/2} n^{-1/2} \text{ cm} \]

\[ \lambda_D / r_{L,e} = 220 B n^{-1/2} \]

- self-shielding distance, plasma charges “screen out” test charge

- collective behaviour of particles, works if \( \lambda_D \ll L \)
2) **plasma physics**

- Debye number, Debye sphere: \( N_D = n \frac{4 \pi}{3} \lambda_D^3 \)
  - for \( N_D >> 1 \): collective behaviour; for \( N_D < 1 \): independent particles

- plasma parameter: \( \Lambda \equiv 1/N_D \)

- ionisation degree = relative number of ions and atoms: \( \zeta = n_i/n_a \)
  - depends on ionizing processes
  - in thermodynamic equilibrium: only temperature-dependent:
  \[
  \frac{n_i}{n_a} \sim g_e \exp\left(-\frac{\Phi_i}{kT}\right)
  \]

  Saha equation: \[
  \frac{\zeta^2}{1-\zeta} \sim \frac{(kT m_e)^{3/2}}{n h^3} \exp\left(\Phi_i / kT\right)
  \]

  - \( \zeta \sim 0.01\% \) sufficient to behave as plasma

- mean free path: \( \lambda = \nu_{\text{thermal}} / \nu_{\text{coll}} \), typical distance between collisions
  - collision-dominated plasma for \( \lambda \ll L \), typically \( \nu_{\text{coll}} \sim \omega_p (\ln \Lambda / \Lambda) \)
  - collisionless plasma for \( \lambda > L \), e.g. coupling by magnetic field collisions
  - help establishing Boltzmann distribution

  \[
  \lambda [cm] = 1.44 \times 10^{13} (\ln \Lambda) n/(kT_e)^2 [eV, cm^{-3}]
  \]
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MHD theory

2) plasma physics

-> summary of parameters of different plasmas
2) plasma physics

-> summary of parameters of different plasmas

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<tr>
<th>Plasma</th>
<th>$n_e$ (m$^{-3}$)</th>
<th>$T$ (K)</th>
<th>$B$ (T)</th>
<th>$\lambda_D$ (m)</th>
<th>$N_D$</th>
<th>$\omega_p$ (s$^{-1}$)</th>
<th>$\nu_{ee}$ (s$^{-1}$)</th>
<th>$\omega_c$ (s$^{-1}$)</th>
<th>$r_L$ (m)</th>
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<tr>
<td>Gas discharge</td>
<td>$10^{16}$</td>
<td>$10^4$</td>
<td>—</td>
<td>$10^{-4}$</td>
<td>$10^4$</td>
<td>$10^{10}$</td>
<td>$10^5$</td>
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<td>$10^{-5}$</td>
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<td>$10^{-1}$</td>
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<td>$10^4$</td>
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<tr>
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<td>1</td>
<td>$10^{18}$</td>
<td>$10^{16}$</td>
<td>—</td>
<td>—</td>
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<tr>
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<td>$10^{-9}$</td>
<td>10</td>
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<td>$10^{-6}$</td>
<td>$10^2$</td>
<td>$10^4$</td>
</tr>
<tr>
<td>Interstellar medium</td>
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<td>$10^{-10}$</td>
<td>10</td>
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<td>10</td>
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</tr>
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<td>Intergalactic medium</td>
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<td>—</td>
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2) **plasma physics**: fluid model, kinetic theory

- plasma physics: **closure of Maxwell equations** by expressions for charge density $\rho_c$ and electric current density $j$
  
  in terms of $E$ and $B$

  by microscopic distribution functions for each plasma species

- define $F_s(r, v, t)$ as **microscopic phase-space density** of plasma species $s$
  
  near point $(r,v)$ at time $t$. $F_s$ normalized to particle density in coordinate space: $\int F_s(r, v, t) \, d^3v = n_s(r, t)$,

- phase space conservation requires: $\frac{\partial F_s}{\partial t} + \mathbf{v} \cdot \nabla F_s + a_s \cdot \nabla_v F_s = 0$,
  
  while $a_s = \frac{e_s}{m_s} (E + \mathbf{v} \times B)$ is acceleration of species $s$ in $B$ and $E$

- averaging over ensemble: $f_s = < F_s >_{\text{ensemble}}$ (a average plus collision operator):

  \[
  \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \bar{a}_s \cdot \nabla_v f_s = C_s(f). \quad \rightarrow \quad \text{Cs extremely complicated}
  \]

  \[
  \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \bar{a}_s \cdot \nabla_v f_s = 0. \quad \rightarrow \quad \text{for simplicity (Vlasov equation)}
  \]
MHD theory

2) plasma physics: kinetic theory, moments of distribution function

density ($1^{\text{st}}$ order): \( n_s(r,t) = \int f_s(r,v,t) \, d^3v, \)

flux density ($1^{\text{st}}$ order): \( n_s V_s(r,t) = \int v f_s(r,v,t) \, d^3v. \quad V_s \text{ is flow velocity} \)

charge density ($1^{\text{st}}$ order): \( \sum_s e_s n_s, \quad \text{electric current density ($1^{\text{st}}$ order):} \quad \sum_s e_s n_s V_s. \)

stress tensor, momentum flow ($2^{\text{nd}}$ order): \( P_s(r,t) = \int m_s vv f_s(r,v,t) \, d^3v. \)

energy flux density ($3^{\text{rd}}$ order): \( Q_s(r,t) = \int \frac{1}{2} m_s v^2 v f_s(r,v,t) \, d^3v. \)

heat flux density (rest frame): \( q_s(r,t) = \int \frac{1}{2} m_s w_s^2 w_s f_s(r,v,t) \, d^3v. \)

pressure tensor (rest frame): \( p_s(r,t) = \int m_s w_s w_s f_s(r,v,t) \, d^3v, \quad w_s \equiv v - V_s, \)

moments in diff. frames: \( Q_s = q_s + p_s \cdot V_s + \frac{3}{2} p_s V_s + \frac{1}{2} m_s n_s V_s^2 V_s. \)

similar for collision operator ...
2) plasma physics: moments of kinetic equation, fluid equations

For each species -> obtain fluid equations by taking moments of ensemble-avaraged kinetic equation

$$\frac{\partial f_s}{\partial t} + \nabla \cdot (\mathbf{v} f_s) + \nabla \cdot (a_s f_s) = C_s(f).$$

continuity equation:

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{V}_s) = 0.$$ 

momentum conservation:

$$\frac{\partial (m_s n_s \mathbf{V}_s)}{\partial t} + \nabla \cdot \mathbf{P}_s - e_s n_s (\mathbf{E} + \mathbf{V}_s \times \mathbf{B}) = \mathbf{F}_s.$$ 

density cons.:

$$\frac{\partial}{\partial t} \left( \frac{3}{2} p_s + \frac{1}{2} m_s n_s V_s^2 \right) + \nabla \cdot \mathbf{Q}_s - e_s n_s \mathbf{E} \cdot \mathbf{V}_s = W_s + \mathbf{V}_s \cdot \mathbf{F}_s.$$ 

-> fluid equations by re-arrangement using pressure tensor,
heat flux density etc .... ---------------> e.g. $$\frac{dn_s}{dt} + n_s \nabla \cdot \mathbf{V}_s = 0,$$
-> closure of equations: express viscosity tensor, heat flux
density, collisional terms in terms of density, velocity, pressure ... 
-> hydrodynamic equations (for each species)
3) MHD equations

-> derived from two-fluid plasma equations under certain simplifications:
   -> \( m_i \gg m_e \); \( v_i \sim v_e \sim v_\text{thermal} \); charge neutrality

-> merge two-fluid equations to get one-fluid equation

example: velocities

\[
\mathbf{V} = \frac{m_i \mathbf{V}_i + m_e \mathbf{V}_e}{m_i + m_e}, \quad \mathbf{j} = -n_e \mathbf{U}, \quad \mathbf{U} = \mathbf{V}_e - \mathbf{V}_i
\]

-> \( \mathbf{V}_i \sim \mathbf{V} + O(m_e/m_i) \), \( \mathbf{V}_e \sim \mathbf{V} - \frac{\mathbf{j}}{n_e} + O\left(\frac{m_e}{m_i}\right) \).

-> from that and

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{V}_e) \quad \text{and} \quad \frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{V}_i) \quad \text{and} \quad \nabla \cdot \mathbf{j} = 0
\]

follows continuity equation:

\[
\frac{dn}{dt} + n \nabla \cdot \mathbf{V} = 0, \quad \frac{\partial}{\partial t} \equiv \partial/\partial t + \mathbf{V} \cdot \nabla
\]

-> similar for equation of motion (add two-fluid equations, total pressure \( p=p_e+p_i \))

\[
m_i n \frac{d\mathbf{V}}{dt} + \nabla p - \mathbf{j} \times \mathbf{B} \approx 0.
\]
Outflows & Jets: Theory & Observations

MHD theory

3) MHD equations

One-fluid equations + Maxwell equations; resistive; eq. of state needed for closure

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho \vec{v}) &= 0 \\
\rho (\partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v}) + \nabla P + \rho \nabla \Phi - \vec{j} \times \vec{B} &= 0 \\
\rho (\partial_t e + (\vec{v} \cdot \nabla) e) + P(\nabla \cdot \vec{v}) - \eta_D |\vec{j}|^2/c^2 &= 0 \\
\partial_t \vec{B} &= \nabla \times (\vec{v} \times \vec{B} - \eta_D \vec{j}/c) \\
\nabla \cdot \vec{B} &= 0, \quad \nabla \times \vec{B} = 4\pi \vec{j}/c
\end{align*}
\]

"no monopoles"  Ampere's law  induction equation

plus: equation of state, eg. polytropic or isothermal gas
Outflows & Jets: Theory & Observations

MHD theory

3) MHD equations, flux freezing

Alfven's theorem (1943): “In a perfectly conducting fluid, magnetic field lines move with the fluid: field lines are "frozen" into the plasma.”

-> A motion along magnetic field lines does not change the field, motions transverse to the field carry the field with them.

Integrate induction equation $\frac{\partial B}{\partial t} = \nabla \times (v \times B)$ with Gauss' theorem $\int_V \nabla \cdot A \, dV = \int_S A \cdot dS$, (\(S\) is closed surface enclosing volume \(V\)) and with Stokes' theorem $\int_S \nabla \times A \cdot dS = \int_C A \cdot dl$, (\(C\) is a closed curve around the open surface \(S\); \(dS = \hat{n}dS\) with the outward unit normal \(\hat{n} \))

(i) Since for all time $\nabla \cdot B = 0$ $\Rightarrow 0 = \int_V \nabla \cdot B \, dV = \int_S B \cdot dS$, $\forall t$, (closed surface \(S\))

(ii) Time behaviour of the magnetic flux $\Phi$ through closed curve \(C\), around an open surface \(S_1\):

$$\Phi = \int_{S_1} B(r, t) \cdot dS.$$ 

$\Phi$ changes in time since $B = B(t)$ and since curve $C$ changes in response to plasma motions.
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MHD theory

3) MHD equations, flux freezing
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MHD theory

3) MHD equations, flux freezing

(iii) curve C moves with the fluid to curve C' within δt. Motion of surface enclosed by C to surface enclosed by C' generates volume V enclosed by surface S.

(iv) total flux through closed surface S in (iii): At time t+δt, when B(r, t+δt), we have

\[ 0 = \int_{\text{closed} S} B(r, t+\delta t) \cdot dS = \int_{\text{top}} B(r, t+\delta t) \cdot dS + \int_{\text{bottom}} B(r, t+\delta t) \cdot dS + \int_{\text{side}} B(r, t+\delta t) \cdot dS, \]

where

\[ = \int_{C'} B(r, t+\delta t) \cdot dS - \int_{C} B(r, t+\delta t) \cdot dS + \int_{\text{side}} B(r, t+\delta t) \cdot dS. \]

(v) Consider contribution to the total flux from curved side. A small element of length on the curve C traces out the shaded region. Then dS is given by the outward normal, \( \hat{n} \) times the area of shaded region. This area is approximately the area of parallelogram with sides dl and vδt. Hence, on the side dS = dl x \( \hat{n} \) δt. Thus,

\[ 0 = \int_{C'} B(r, t+\delta t) \cdot dS - \int_{C} B(r, t+\delta t) \cdot dS + \int_{C} B(r, t+\delta t) \cdot dl \times v \delta t. \]

\[ \int_{C'} B(r, t+\delta t) \cdot dS = \int_{C} B(r, t+\delta t) \cdot dS - \delta t \int_{C} B(r, t+\delta t) \cdot dl \times v, \]

-> flux through curve C' at t+δt is equal to flux through curve C minus contribution from sides.
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MHD theory

3) MHD equations, flux freezing

(vi) Change in flux in time is difference \( \Phi(t+\delta t) - \Phi(t) \):
\[
\delta \Phi = \Phi(t + \delta t) \text{ through } C' - \Phi(t) \text{ through } C,
\]
\[
= \int_{C'} B(r, t + \delta t) \cdot dS - \int_C B(r, t) \cdot dS.
\]

With *** (2.41):
\[
\delta \Phi = \int_C \left[ B(r, t + \delta t) - \int_C B(r, t) \right] \cdot dS - \delta t \int_C B(r, t + \delta t) \cdot dl \times v.
\]

Small \( \delta t \) -> approximate integrand in surface integral: \( B(r, t + \delta t) - \int_C B(r, t) \rightarrow \delta t \partial B / \partial t \)
\[
\rightarrow \delta \Phi = \delta t \int_C \frac{\partial B}{\partial t} \cdot dS - \delta t \int_C v \times B \cdot dl. \quad \text{(identity } B \cdot (dl \times v) = v \cdot (B \times dl) = (v \times B) \cdot dl \text{)}
\]

(vii) w/ induction eq.:
\[
\frac{\delta \Phi}{\delta t} = \int_C \nabla \times (v \times B) \cdot dS - \int_C v \times B \cdot dl,
\]
\[
= \int_C v \times B \cdot dl - \int_C (v \times B) \cdot dl, \quad \text{on using Stoke's theorem,}
\]
\[
= 0.
\]

As \( \delta t \rightarrow 0 \), \( \frac{\delta \Phi}{\delta t} \rightarrow \frac{d\Phi}{dt} \), thus \( \Phi \) does not change in time,
\[
\frac{d\Phi}{dt} = \frac{d}{dt} \left\{ \int_C B \cdot dS \right\} = 0,
\]

where \( C \) is any closed contour moving with the fluid.

\( \Rightarrow \) Field lines are frozen into the plasma!
Outflows & Jets: Theory & Observations

MHD theory

MHD waves

-> define dynamical time scales // transport information / energy

-> linearize MHD equations, using $Q \to Q_0 + Q$; $Q_0$: equilibrium quantity, $Q$: perturbed q.

\[
\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \mathbf{V} = 0 \quad \frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{V} \times \mathbf{B}_0) = 0 \\
\frac{\partial}{\partial t} \left( \frac{p}{p_0} - \frac{\Gamma \rho}{\rho_0} \right) = 0 \quad \rho_0 \frac{\partial \mathbf{V}}{\partial t} + \nabla p - \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}_0}{\mu_0} = 0
\]

-> look for wave-like solutions of linearized MHD equations, $Q \sim \exp( k^* r - \omega t )$ :

\[
-\omega \rho + \rho_0 \mathbf{k} \cdot \mathbf{V} = 0 \quad \omega \mathbf{B} + \mathbf{k} \times (\mathbf{V} \times \mathbf{B}_0) = 0 \\
-\omega \left( \frac{p}{p_0} - \frac{\Gamma \rho}{\rho_0} \right) = 0 \quad -\omega \rho_0 \mathbf{V} + \mathbf{k} p - \frac{(\mathbf{k} \times \mathbf{B}) \times \mathbf{B}_0}{\mu_0} = 0
\]

-> substitute into linearized e.o.m. :

\[
\left[ \omega^2 - \frac{(\mathbf{k} \cdot \mathbf{B}_0)^2}{\mu_0 \rho_0} \right] \mathbf{V} \left\{ \left[ \frac{\Gamma p_0}{\rho_0} + \frac{B_0^2}{\mu_0 \rho_0} \right] \mathbf{k} - \frac{(\mathbf{k} \cdot \mathbf{B}_0)}{\mu_0 \rho_0} \mathbf{B}_0 \right\} (\mathbf{k} \cdot \mathbf{V}) - \frac{(\mathbf{k} \cdot \mathbf{B}_0) (\mathbf{V} \cdot \mathbf{B}_0)}{\mu_0 \rho_0} \mathbf{k}.
\]
Outflows & Jets: Theory & Observations

MHD theory

MHD waves

--> Define e.g. $\mathbf{B}_0 \parallel \mathbf{e}_z$, wave-vector $\mathbf{k}$ in x-z plane, $\theta$ is angle between $\mathbf{B}_0$ and $\mathbf{k}$

--> eigenvalue equation:

$$
\begin{pmatrix}
\omega^2 - k^2 V_A^2 - k^2 V_S^2 \sin^2 \theta & 0 & -k^2 V_S^2 \sin \theta \cos \theta \\
0 & \omega^2 - k^2 V_A^2 \cos^2 \theta & 0 \\
-k^2 V_S^2 \sin \theta \cos \theta & 0 & \omega^2 - k^2 V_S^2 \cos^2 \theta
\end{pmatrix}
\begin{pmatrix}
V_x \\
V_y \\
V_z
\end{pmatrix}
= 0.
$$

Alfven speed

$$V_A = \sqrt{\frac{B_0^2}{\mu_0 \rho_0}}$$

and sound speed

$$V_S = \sqrt{\frac{\Gamma p_0}{\rho_0}}$$

--> eigenvalue equation solvable if determinant of square matrix vanishes

--> dispersion relation:

\[
(\omega^2 - k^2 V_A^2 \cos^2 \theta) \left[ \omega^4 - \omega^2 k^2 (V_A^2 + V_S^2) + k^4 V_A^2 V_S^2 \cos^2 \theta \right] = 0.
\]
Outflows & Jets: Theory & Observations

MHD theory

MHD waves

Three independent roots:

1) \( \omega = k V_A \cos \theta \), eigenvector \((0, V_y, 0)\), \( k \cdot \mathbf{V} = 0 \), \( \mathbf{V} \cdot \mathbf{B}_0 = 0 \)

-> shear Alfven wave; no perturbation of plasma density; motion \( \perp \) field

2) \( \omega = k V_+ \), \( \omega = k V_- \),

with \( V_\pm = \left\{ \frac{1}{2} \left[ V_A^2 + V_S^2 \pm \sqrt{(V_A^2 + V_S^2)^2 - 4 V_A^2 V_S^2 \cos^2 \theta} \right] \right\}^{1/2} \).

Note that \( V_+ \geq V_- \), eigenvector \((V_x, 0, V_z)\) \( k \cdot \mathbf{V} \neq 0 \), \( \mathbf{V} \cdot \mathbf{B}_0 \neq 0 \)

-> perturbations in density / pressure, motion \( \parallel \) and \( \perp \) to magnetic field

-> fast magnetosonic wave (2) and slow magnetosonic wave (3)

For limit \( V_A \gg V_S \) (strong field), slow wave dispersion reduces, \( \omega \sim k V_S \cos \theta \).

-> sound wave along magnetic field lines
Phase velocities of 3 MHD waves; low plasma-β with \( V_s < V_A \)
Outflows & Jets: Theory & Observations

MHD theory

Stationary axisymmetric MHD

First (?) presented by Chandrasekhar (1956):

-> Interesting setup for **jets and outflows** -> essential properties drop out naturally

-> Example: derivation for Ferraro's law of isorotation:

1) use cylindrical coordinates (r,φ,z)

2) decompose vectors in **poloidal** (meridional plane), & **toroidal** components (φ-component)

3) stationary Faraday's law: \( \nabla \times \vec{E} = 0 \) --&gt; potential field: \( \vec{E} = \nabla U \)

4) axisymmetry: \( E_\phi = 0 \)

5) infinite conductivity: Ohms law: \( \vec{E} = \frac{1}{\sigma} \vec{v} \times \vec{B} \)

6) since \( E_\phi = 0 \) --&gt; \( \vec{v}_p \times \vec{B}_p = 0 \). or \( \vec{v}_p \parallel \vec{B}_p \) --&gt; \( \vec{v}_p = \kappa(R,Z)\vec{B}_p \)

-> poloidal velocity parallel to poloidal field

7) mass conservation & stationarity --&gt; \( \nabla (\rho \vec{v}_p) = 0 \)

8) for \( \kappa(R,Z) \) --&gt; \( 0 = \nabla (\rho \kappa \vec{B}_p) = B_p \cdot \nabla (\rho \kappa) \) --&gt; \( \eta \equiv \rho \kappa \) conserved along field lines

9) introduce **magnetic flux function**:

\[
\Psi(R,Z) = \frac{1}{2\pi} \int \vec{B}_p \cdot d\vec{A} = \frac{1}{2\pi} \int B_z R d\phi dR,
\]
Stationary axisymmetric MHD

-> exemple: derivation for Ferraro's law of isorotation

9) Introduce magnetic flux function:
\[ \Psi(R, Z) = \frac{1}{2\pi} \int \vec{B}_p \cdot d\vec{A} = \frac{1}{2\pi} \int B_z R \, d\phi \, dR, \]

10) We have further
\[ \vec{v} \times \vec{B} = \vec{v}_\phi \times \vec{B}_p + \vec{v}_p \times \vec{B}_\phi = \frac{1}{R} \left( v_\phi - \frac{\eta}{\rho} B_\phi \right) \nabla \Psi \]

11) With MHD condition 5) and Faraday's law:
\[ 0 = \nabla \times \vec{E} = \nabla \Psi \times \nabla \left( \frac{1}{R} \left( v_\phi - \frac{\eta}{\rho} B_\phi \right) \right) \]

12) Thus the quantity
\[ \Omega_F \equiv \left( v_\phi - \frac{\eta}{\rho} B_\phi \right) / R \]

is conserved along the field line

-> Ferraro's law of isorotation, iso-rotation parameter, “angular velocity of field line”
Outflows & Jets: Theory & Observations

MHD theory

Stationary axisymmetric MHD

Conservation laws of stationary MHD:

Mass flow rate per flux surface:
\[
\eta(\Psi) \equiv \rho \frac{v_p}{B_p} \text{sgn}(\vec{v}_p \cdot \vec{B}_p)
\]

\[
\dot{M}(\Psi + \Delta \Psi) - \dot{M}(\Psi) = \int_{\Psi}^{\Psi + \Delta \Psi} \rho \vec{v}_p \cdot d\vec{A} = \int_{\Psi}^{\Psi + \Delta \Psi} \eta \vec{B}_p \cdot d\vec{A}
\]

Field line iso-rotation:
\[
\Omega_F(\Psi) \equiv \frac{1}{R} \left( v_\phi - \frac{\eta(\Psi)}{\rho} B_\phi \right)
\]

Energy conservation:
\[
E(\Psi) = \frac{v^2}{2} - \frac{RB_\phi \Omega_F(\Psi)}{4\pi \eta(\Psi)}
\]

Angular momentum conservation:
\[
L(\Psi) = R^2 \Omega_F(\Psi) - \frac{RB_\phi}{4\pi \eta(\Psi)}
\]
Outflows & Jets: Theory & Observations
Stationary MHD - the solar wind

Example: Parker wind

Parker's (1958) prediction: solar corona not in static equilibrium \(\rightarrow\) expansion
(later confirmed by satellite missions)

Parker model of solar wind: stationary, spherically symmetric, hydrodynamic, isothermal

\(-\) mass conservation:
\[
\nabla \cdot (\rho \mathbf{v}) = 0
\]

\(-\) momentum conservation:\n\[
\rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P + \rho \mathbf{g}
\]

\(-\) assumptions:\n\[
P = \rho R T, \ T = T_0, \ \mathbf{v} = \mathbf{v} \hat{r}, \quad g = -GM_0/r^2
\]

\(-\) mass conservation:\n\[
\frac{d}{dr}(r^2 \rho v) = 0 \quad \rightarrow \quad r^2 \rho v = \text{const.}
\]

\(-\) radial momentum conservation:\n\[
\rho v \frac{dv}{dr} = -\frac{dP}{dr} - \frac{GM_0 \rho}{r^2}
\]

\(-\) applying isothermal sound speed:\n\[
c_s \equiv (P/\rho)^{1/2}, \quad P = c_s^2 \rho
\]

\(-\) \[
\rho v \frac{dv}{dr} = -c_s^2 \frac{d\rho}{dr} - \frac{GM_0 \rho}{r^2}
\]
Outflows & Jets: Theory & Observations

Stationary MHD - the solar wind

Example: Parker wind

-> applying mass conservation

-> wind equation

-> integrate analytically / numerically:

-> critical wind solution:

\[ v \frac{dv}{dr} = -c_s^2 r^2 v \frac{d}{dr} \left( \frac{1}{r^2 v} \right) - \frac{GM_o}{r^2} \]

\[ (1 - \frac{c_s^2}{v^2}) v \frac{dv}{dr} = 2 \frac{c_s^2}{r^2} (r - r_s) \]

-> critical point:

\[ r_s = \frac{GM_o}{c_s^2} \]

-> \( v = c_s \)

-> critical point = sonic point

-> solar parameters:

-> \( T = 10^6 \) (corona)

\( c_s = 100 \) km/s

\( r_s = 10 \) ro

\( v(1AU) = 310 \) km/s (pred.)

\( v(1AU) = 320 \) km/s (obs.)
Outflows & Jets: Theory & Observations

Stationary MHD - the solar wind

Example: Solar wind / Weber & Davis (1967)

-> magnetized solar wind; magnetic wind equation: Alfven Mach number $M_A = v / v_A$

-> stellar rotation essential; magnetic field removes angular momentum -> stellar braking

-> radial momentum conservation:

$$\frac{d}{dr} \left\{ \frac{1}{2} u^2 + \frac{\gamma}{\gamma - 1} \frac{\rho}{\rho_a} \left( \frac{\rho}{\rho_a} \right)^{\gamma - 1} - \frac{GM_\odot}{r} \right\} = \frac{v_\phi^2}{r} - \frac{1}{8 \pi \rho r^2} \frac{d}{dr} (rB_\phi)^2$$

-> magnetic wind equation:

$$\frac{du}{dr} = \frac{u}{r} \left\{ \left( \frac{2 \gamma \rho}{\rho_a M_A^{2(\gamma - 1)}} - \frac{GM_\odot}{r} \right) (M_A^2 - 1)^3 

+ \Omega^2 r^2 \left( \frac{u}{u_a} - 1 \right) \left[ (M_A^2 + 1) \frac{u}{u_a} - 3 M_A^2 + 1 \right] \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} 

-> additional critical points / singularities:

-> slow magnetosonic point / Alfven point / fast magnetosonic point

determine critical solution:

-> $v = v_{sm}$ at $r = r_{sm}$; $v = v_A$ at $r = r_A$; $v = v_{fm}$ at $r = r_{fm}$
Outflows & Jets: Theory & Observations

Stationary MHD - relativistic wind

Example: MHD wind in Kerr metric

--> relativistic treatment of motion and metric:
relativistically defined velocity: \( u_p = \gamma v_p / c \),

--> wind equation:
\[
\frac{u_p^2}{\sigma_m} + 1 = \left( \frac{E}{\mu} \right)^2 \frac{k_0 k_2 + \sigma_m 2k_2 M^2 - k_4 M^4}{(k_0 + \sigma_m M^2)^2},
\]

metric:
\[
\begin{align*}
k_0 &= g_{33} \Omega_F^2 + 2g_{03} \Omega_F + g_{00}, \\
k_2 &= 1 - \Omega_F \frac{L}{E}, \\
k_4 &= - \left( g_{33} + 2g_{03} \frac{L}{E} + g_{00} \frac{L^2}{E^2} \right) / \left( g_{03}^2 - g_{00} g_{33} \right)
\end{align*}
\]

--> 3 critical points
--> (highly) relativistic velocities
--> MHD condition applicable ??
Outflows & Jets: Theory & Observations

Stationary MHD - stellar winds

Axisymmetric structure of MHD jets:

- dynamics along a given field line by wind equation
- structure magnetic field:
  - force-balance across the field / flux surfaces
  - project eq. of motion to magnetic surfaces $a(r,z)$:
    - consider $\phi$-component of Ampere's law
    - take current density from eq. of motion
  - Grad-Shafranov (GS) equation: (curvature $\Psi$)

\[
\left(1 - \frac{\rho}{\rho_A}\right) v_P^2 \frac{d\psi}{ds} = \frac{1}{\rho |\nabla a|} \cdot \nabla \left( \frac{|\nabla a|^2}{2\mu_0 r^2} + Q \rho^\gamma \right) + \frac{\nabla a}{|\nabla a|} \cdot \nabla \Phi_G \\
- \left( L - \frac{I}{\alpha} \right)^2 \frac{1}{r^3 |\nabla a|} \frac{\partial a}{\partial r} + \frac{1}{\rho r^2 |\nabla a|} \cdot \nabla \left( \frac{\mu_0 I^2}{2} \right). \tag{12}
\]

- r.h.s.: poloidal magnetic pressure gradient, gas pressure gradient, gravity, centrifugal force, hoop stress
- GS contains dynamical parameters (density, velocity, ...)
- to be calculated by wind equation
- iterative procedure for solution (Sakurai 1985)
Outflows & Jets: Theory & Observations

**MHD theory -- simulations**

**Time-dependent solutions**

of MHD equations

by numerical simulations

- Numerical MHD codes:
  ZEUS, Flash, Pluto, Nirvana ...

- apply astrophysical boundary conditions (disk/stellar magnetic field, mass flow rates ....)

- some advantages:
  - time-dependent evolution
  - no need to search for critical solutions
  - 3D solutions possible
  - inclusion of more physics “simple”
    (radiation losses, turbulent viscosity, resistivity ...)

- some difficulties:
  - dynamic range (strong density contrast, strong gradients)
  - computer power limited (grid size, time resolution)

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho \vec{v}) &= 0 \\
\rho (\partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v}) + \nabla P + \rho \nabla \Phi - \vec{j} \times \vec{B} &= 0 \\
\rho (\partial_t e + (\vec{v} \cdot \nabla) e) + P (\nabla \cdot \vec{v}) - \eta_D |\vec{j}|^2 / c^2 &= 0 \\
\partial_t \vec{B} &= \nabla \times (\vec{v} \times \vec{B} - \eta_D \vec{j} / c) \\
\nabla \cdot \vec{B} &= 0, \quad \nabla \times \vec{B} = 4\pi \vec{j} / c
\end{align*}
\]
Outflows & Jets: Theory & Observations

10.10 Introduction & Overview ("H.B." & C.F.)
17.10 Definitions, parameters, basic observations (H.B.)
24.10 Basic theoretical concepts & models I (C.F.): Astrophysical models, MHD
31.10 Basic theoretical concepts & models II (C.F.): MHD, derivations, applications

07.11 Observational properties of accretion disks (H.B.)
14.11 Accretion disk theory and jet launching (C.F.)
21.11 Outflow-disk connection, outflow entrainment (H.B.)
28.11 Outflow-ISM interaction, outflow chemistry (H.B.)
05.12 Theory of outflow interactions; Instabilities (C.F.)
12.12 Outflows from massive star-forming regions (H.B.)
19.12 Radiation processes - 1 (C.F.)

26.12 and 02.01 Christmas and New Year's break
09.01 Radiation processes - 2 (H.B.)
16.01 Observations of AGN jets (C.F.)
23.01 Some aspects of AGN jet theory (C.F.)
30.01 Summary, Outlook, Questions (H.B. & C.F.)