A Quick Review of Cosmology: The Geometry of Space, Dark Matter, and the Formation of Structure

Hans-Walter Rix
Max-Planck-Institute for Astronomy

Cosmology:

a) Try to understand the origin, the structure, mass-energy content and the evolution of the universe as a whole.

b) To understand the emergence of structures and objects ranging from scales as small as stars (10^{10} m) to scales much larger than galaxies ($\sim 10^{26}$ m) through gravitational self-organization.

Textbooks: Peacock, Padmanaban

1) Elements of standard world model

a) Averaged over sufficiently large scales (≥ 1000 Mpc), the universe is approximately homogeneous and isotropic ("cosmological principle").

b) The universe is expanding so that the distance (to defined precisely later) between <u>any</u> two widely separated points increases as: $dl/dt = H(t) * l$

c) Expansion dynamics of the universe are determined by the mass and energy content (General Relativity).

d) universe had early hot and dense state: big bang

e) On small scales (≤ 100 Mpc), a great deal of structure has formed, mostly through "gravitational self-organization": stars, galaxy clusters.

2) Homogeneous Cosmology

Starting point:

What is the universe expanding into?

- The observable universe is a lower dimensional sub-space expanding within a higher dimensional space.

OR

- We can describe the expanding 3D universe without reference to higher dimensions (has proven more useful prescription).

Note: Here, we restrict ourselves to the macrosopic description of curved space; all issues of quantum gravity (string theory) will be left out.

2.1. The Robertson Walker Metric

$$\Rightarrow ds^2 = dt^2 - \frac{a^2(t)}{c^2}\left[dr^2 + R^2 \sin^2\left(\frac{r}{R}\right)\cdot\left(d\vartheta^2 + \sin^2\vartheta\delta\varphi^2\right)\right]$$

R = present-day curvature
r = comoving radial coordinates
a(*t*) = expansion or scale factor
NB: a(t) subsumes all time dependence that is compatible with the cosmological principle.

- So far, the evolution of *a*(*t*) is unspecified, i.e. no physics yet, just math.
- General relativity will determine *a*(*t*) as a function of the mass (energy) density **and** link it to *R*!
- The "distances" *r* are not observable, just coordinate distances.

2.2.) General Relativity + Robertson Walker Metric → Friedman Equation

Demanding isotropy and homogeneity, the time dependent solution family to Einstein's field equation is quite simple:

$$\frac{\dot{a}(t)}{a(t)} = H_0 \cdot E(z) = H_0 \cdot \sqrt{\Omega (1+z)^3 + \Omega_R (1+z) + \Omega_\Lambda}$$

with $\Omega \equiv \frac{8\pi G \rho_0}{3H_0}$, $\Omega_R = (H_0\, a_0\, R)^{-2}$, H_0 = const, and $a=(1+z)^{-1}$

$\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$ and $\Omega_{mass\_and\_radiation} + \Omega_R + \Omega_\Lambda = 1$.

a) $\rho_{mass} \sim a^{-3}$

b) $\rho_{radiation} \sim a^{-4}$

c) $\rho_{vac} = const. \Leftrightarrow \Omega_{vac} \equiv \Lambda c^2 / 3H_0^2$

2.3.) Distance Measure(s) in Cosmology

- In curved and expanding space:
  - app. size $\neq \frac{1}{\text{distance}}$
  - luminosity $\neq \frac{1}{\text{distance}^2}$
  - Is there a unique measure of distance, anyway?

- Some observables do not depend on the expansion history, $a(t)$, which we don't know (yet)!

present epoch Hubble constant $$H_0 = 100\,h\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$$

Hubble time $$t_\mathrm{H} \equiv \frac{1}{H_0} = 9.78 \times 10^9\,h^{-1}\ \mathrm{yr} = 3.09 \times 10^{17}\,h^{-1}\ \mathrm{s}$$

Hubble radius/distance $$D_\mathrm{H} \equiv \frac{c}{H_0} = 3000\,h^{-1}\ \mathrm{Mpc} = 9.26 \times 10^{25}\,h^{-1}\ \mathrm{m}$$

"Omega Matter" $$\Omega_\mathrm{M} \equiv \frac{8\pi\,G\,\rho_0}{3\,H_0^2}$$

"Omega Lambda" $$\Omega_\Lambda \equiv \frac{\Lambda\,c^2}{3\,H_0^2}$$

"equiv. Omega curvature" $$\Omega_\mathrm{M} + \Omega_\Lambda + \Omega_k = 1$$

redshift $$z \equiv \frac{\nu_\mathrm{e}}{\nu_\mathrm{o}} - 1 = \frac{\lambda_\mathrm{o}}{\lambda_\mathrm{e}} - 1$$

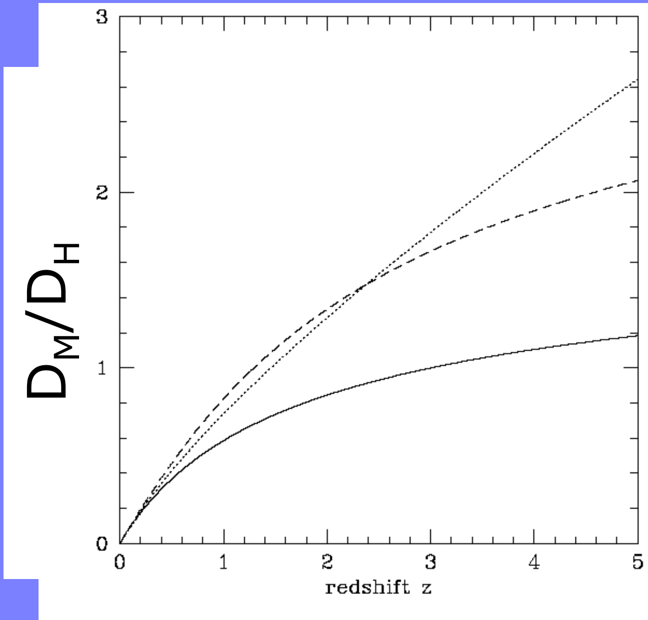
Comoving distance (line-of-sight)

=invariant under expansion

$$E(z) \equiv \sqrt{\Omega_M (1+z)^3 + \Omega_k (1+z)^2 + \Omega_\Lambda}$$

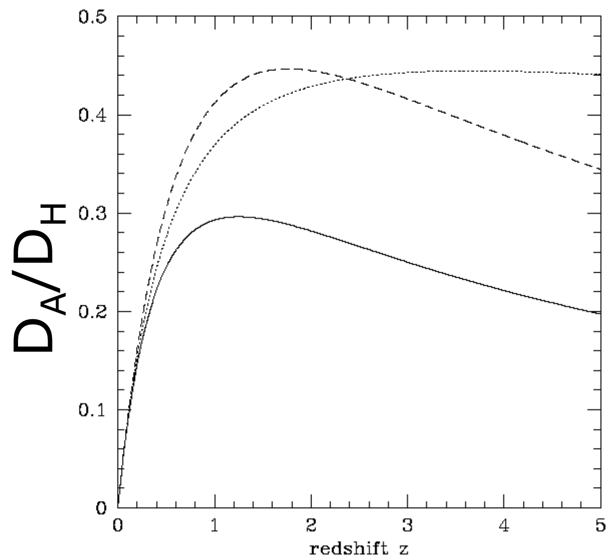
$$H(z) = H_0 \, E(z)$$

$$D_C = D_H \int_0^z \frac{dz'}{E(z')}$$



Comoving distance (transverse)

$$D_M = \begin{cases} D_H \frac{1}{\sqrt{\Omega_k}} \sinh\left[\sqrt{\Omega_k}\, D_C/D_H\right] & \text{for } \Omega_k > 0 \\ D_C & \text{for } \Omega_k = 0 \\ D_H \frac{1}{\sqrt{|\Omega_k|}} \sin\left[\sqrt{|\Omega_k|}\, D_C/D_H\right] & \text{for } \Omega_k < 0 \end{cases}$$



Angular diameter distance

$$D_A = \frac{D_M}{1+z}$$

= phys. size of object / observed angular size

Luminosity distance

The *luminosity distance* D_L is defined by the relationship between bolometric (ie, integrated over all frequencies) flux S and bolometric luminosity L:

$$D_L \equiv \sqrt{\frac{L}{4\pi\, S}}$$

$$D_L = (1+z)\, D_M = (1+z)^2\, D_A$$

$$S_\nu = (1+z)\, \frac{L_{(1+z)\nu}}{L_\nu}\, \frac{L_\nu}{4\pi\, D_L^2} \qquad S_\lambda = \frac{1}{(1+z)}\, \frac{L_{\lambda/(1+z)}}{L_\lambda}\, \frac{L_\lambda}{4\pi\, D_L^2}$$

$$DM \equiv 5 \log\left(\frac{D_L}{10\ \mathrm{pc}}\right)$$

$$m = M + DM + K$$

K is the k-correction

$$K = -2.5 \log\left[(1+z)\, \frac{L_{(1+z)\nu}}{L_\nu}\right] = -2.5 \log\left[\frac{1}{(1+z)}\, \frac{L_{\lambda/(1+z)}}{L_\lambda}\right]$$

5. The Cosmic Microwave Background : Direct Constraint on the Young Universe

A. Overview

- The universe started from a dense and hot initial state ("Big Bang") . As the universe expands, it cools
$$T(z) \sim \frac{1}{size(z)} \sim 1+z$$
- In the "first three minutes" many interesting phenomena occur: e.g. inflation, the ‚seeding' of density fluctuations and primordial nucleosynthesis.
- As long as (ordinary, baryonic) matter is ionized (mostly H^+ and e^-), it is tightly coupled to the radiation through Thompson scattering (needs free electrons!).
  - Radiation has blackbody spectrum
$$I_\nu = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$
  - Mean free path of the photon is small compared to the size of the universe.

- We know from present-day measurements that

$$\frac{N_{phot}}{N_{baryon}} \sim 4 \cdot 10^7$$

  - As long as $T_{radiation} \geq 4000$ K, there are enough photons with $h\nu \geq 13.6$ *eV* to re-ionize virtually every neutral H atom.

- At later epochs (lower $T_{radiation}$), the H^+ and e^- (re)-combine

  - No more Thompson scattering.
  - Photons stream freely, portraying a map of the "last scattering surface", like the surface of a cloud.

B. (Some) Physics of the Microwave Background

When did recombination occur, or what is the redshift of the CMB radiation?

- $T_{recomb} \approx 3500$ K
  - Note that $T_{recomb} \ll 13 \cdot 6$ eV, because only 10^{-7} of the photons need E ~ 13.6 eV.

  $T_{now} \approx 3$ K

$$\Rightarrow \frac{\text{size of universe } (T_{recomb})}{\text{size now}} = \frac{1}{1200}$$

- At that time, the universe was ~ 350,000 years old.

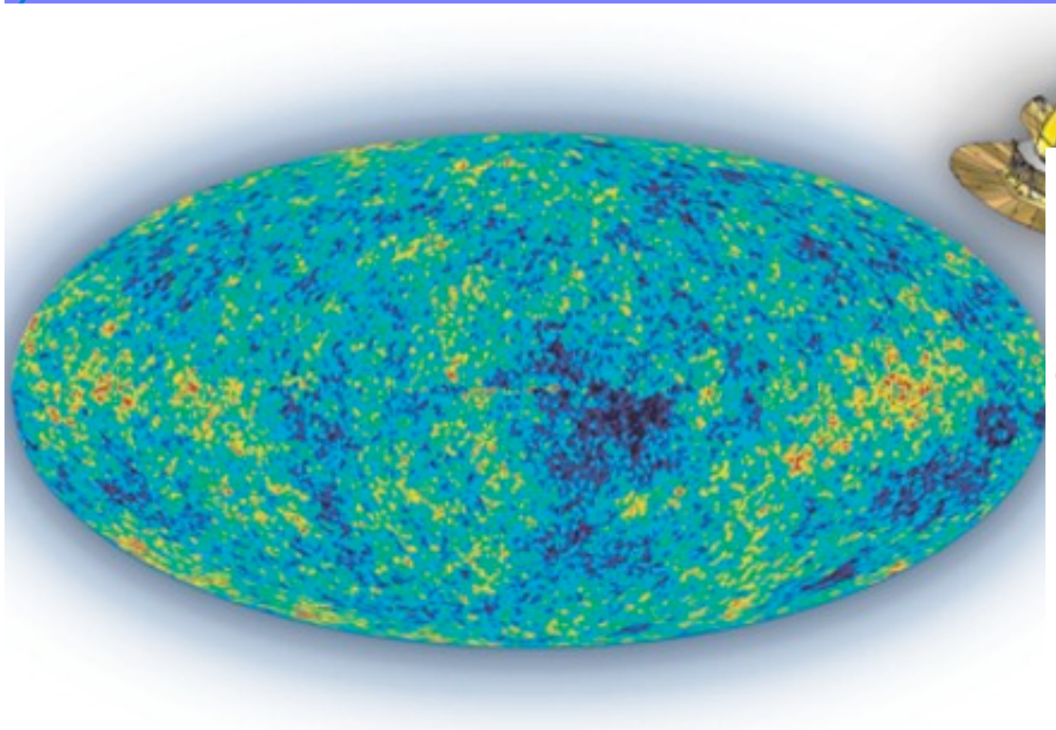
- Only regions with $R < ct_{age}$ can be causally connected.

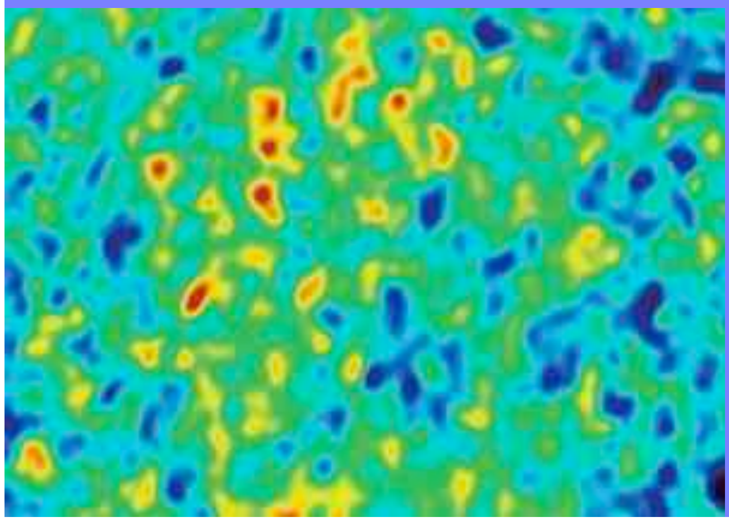
- Such regions appear under an angle $\upsilon \sim 1°$.

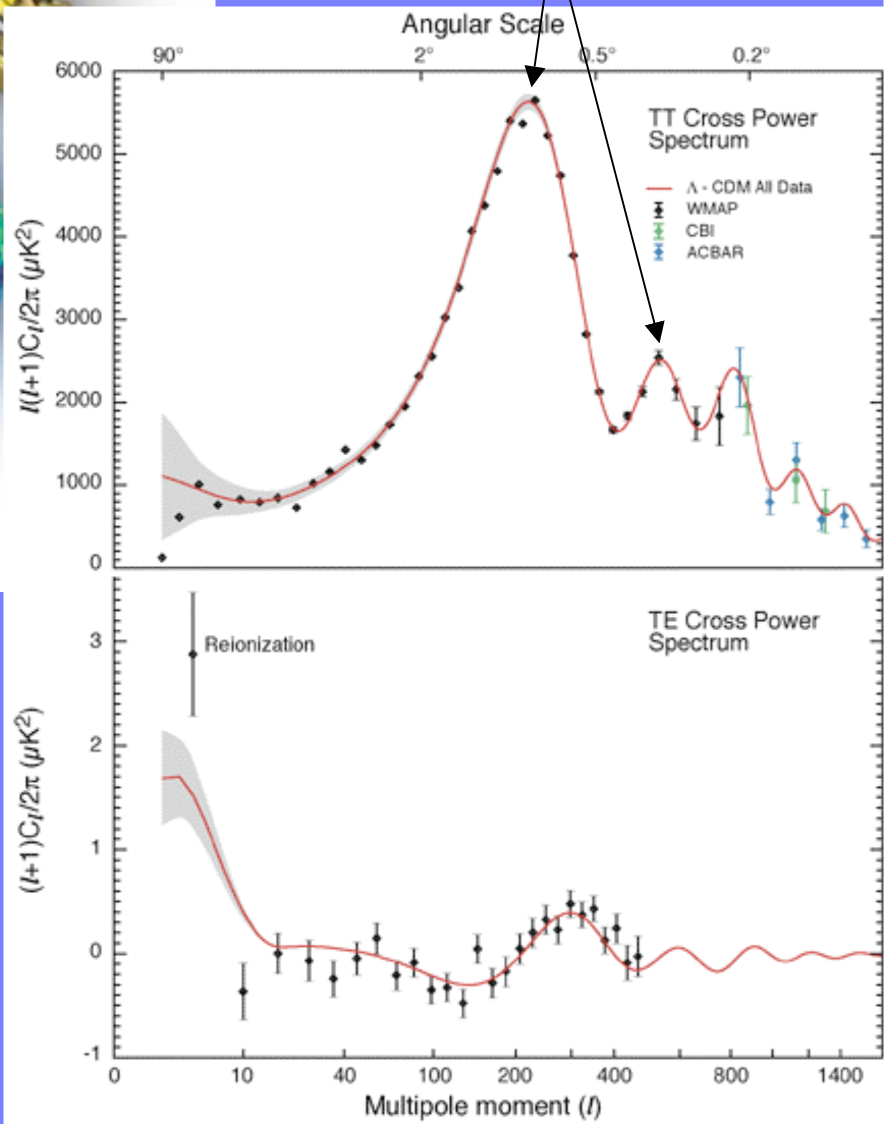
- Therefore, we might expect that the temperature from patches separated by more than ~ 1° is uncorrelated?

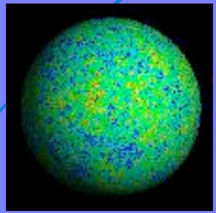
Results of the WMAP Mission

Fluctuations strongest at harmonic “peak” scales



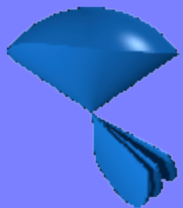






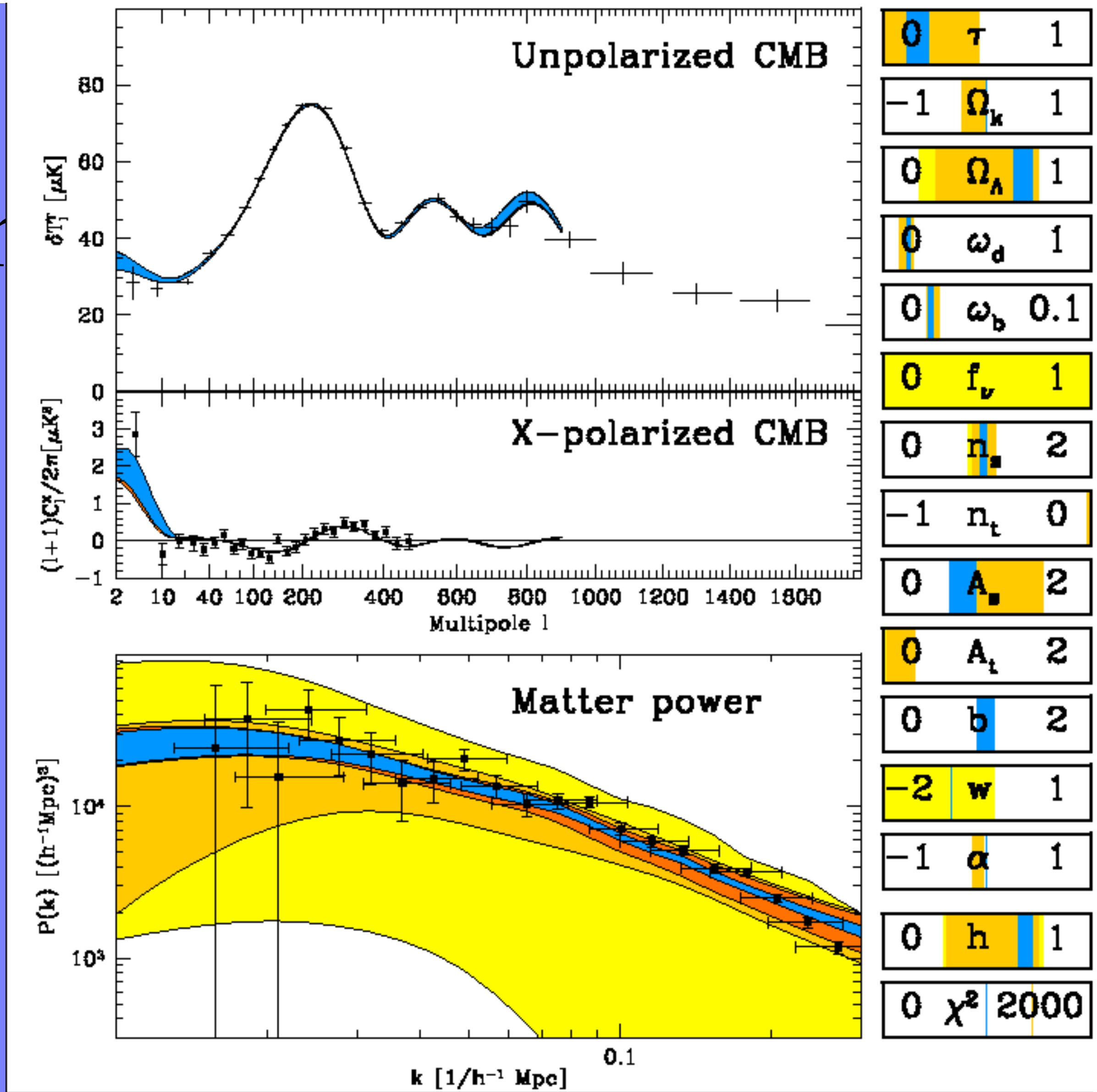
CMB

+

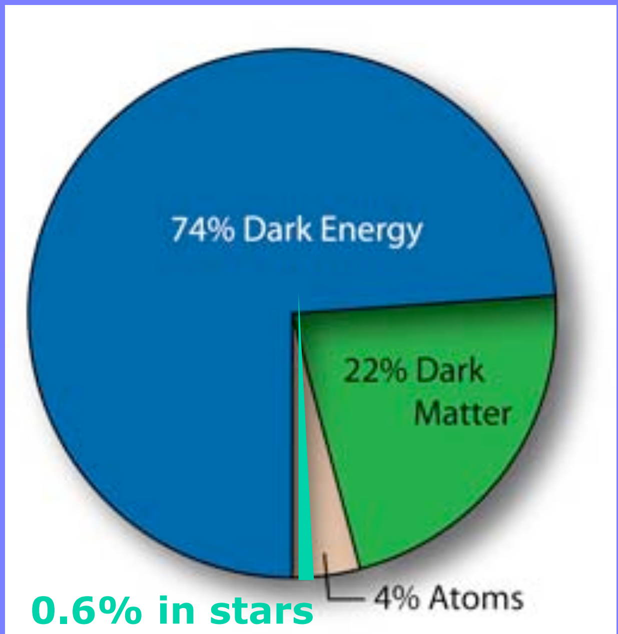


LSS

Cr

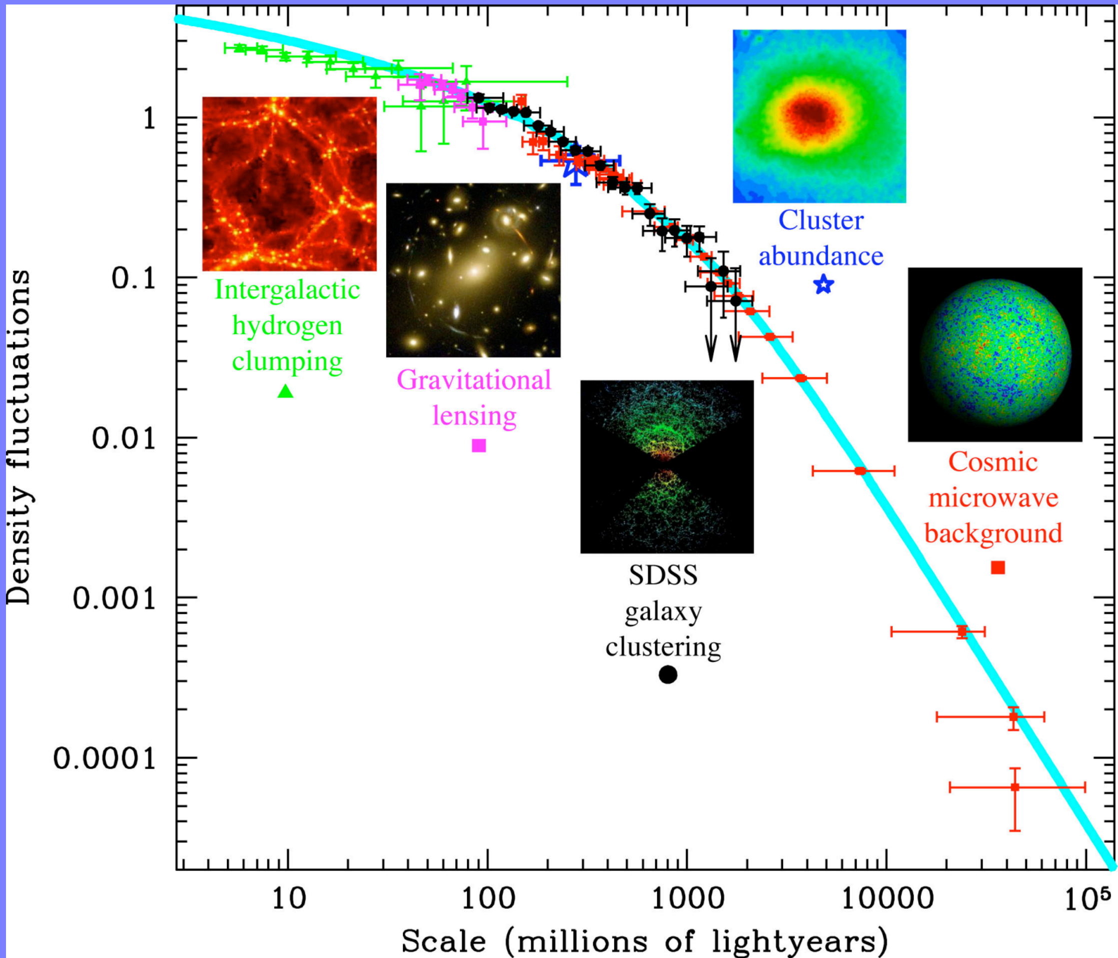


Unpolarized CMB
X−polarized CMB
Matter power
Multipole l
k [1/h⁻¹ Mpc]
0 τ 1
−1 Ω_k 1
0 Ω_Λ 1
0 ω_d 1
0 ω_b 0.1
0 f_ν 1
0 n_s 2
−1 n_t 0
0 A_s 2
0 A_t 2
0 b 2
−2 w 1
−1 α 1
0 h 1
0 χ^2 2000



74% Dark Energy
22% Dark Matter
4% Atoms
0.6% in stars

"Standard Cosmological Model"



- Spergel et al 2003 and 2007

Basic and Derived Cosmological Parameters: Running Spectral Index Model

Parameters	Mean and 68% Confidence Errors
Basic	
Amplitude of fluctuations, A	$0.83^{+0.09}_{-0.08}$
Spectral index at $k = 0.05$ Mpc^{-1}, n_s	0.93 ± 0.03
Derivative of spectral index, $dn_s/d\ln k$	$-0.031^{+0.016}_{-0.018}$
Hubble constant, h	$0.71^{+0.04}_{-0.03}$
Baryon density, $\Omega_b h^2$	0.0224 ± 0.0009
Matter density, $\Omega_m h^2$	$0.135^{+0.008}_{-0.009}$
Optical depth, τ	0.17 ± 0.06
Derived	
Matter power spectrum normalization, σ_8	0.84 ± 0.04
Characteristic amplitude of velocity fluctuations, $\sigma_8 \Omega_m^{0.6}$	$0.38^{+0.04}_{-0.05}$
Baryon density/critical density, Ω_b	0.044 ± 0.004
Matter density/critical density, Ω_m	0.27 ± 0.04
Age of the universe, t_0	13.7 ± 0.2 Gyr
Reionization redshift,[a] z_r	17 ± 4
Decoupling redshift, $z_{\rm dec}$	1089 ± 1
Age of the universe at decoupling, $t_{\rm dec}$	379^{+8}_{-7} kyr
Thickness of surface of last scatter, $\Delta z_{\rm dec}$	195 ± 2
Thickness of surface of last scatter, $\Delta t_{\rm dec}$	118^{+3}_{-2} kyr
Redshift of matter/radiation equality, $z_{\rm eq}$	3233^{+194}_{-210}
Sound horizon at decoupling, r_s	147 ± 2 Mpc
Angular size distance to the decoupling surface, d_A	$14.0^{+0.2}_{-0.3}$ Gpc
Acoustic angular scale,[b] ℓ_A	301 ± 1
Current density of baryons, n_b	$(2.5 \pm 0.1) \times 10^{-7}$ cm^{-3}
Baryon/photon ratio, η	$(6.1^{+0.3}_{-0.2}) \times 10^{-10}$

Note.—Fit to the *WMAP*, CBI, ACBAR, 2dFGRS, and Lyα forest data.

[a] Assumes ionization fraction, $x_e = 1$.

[b] $l_A = \pi d_C / r_s$.

See also Spergel et al 2007 (WMAP 3yr data)

3. The growth of structure: the evolution of density fluctuations

Goal:

Can we explain quantitatively the observed "structure" (galaxy clusters, superclusters, their abundance and spatial distribution, and the Lyman-α forest) as arising from small fluctuations in the nearly homogeneous early universe?

3.1. Linear Theory of Fluctuation Growth

- Growth from $\delta\rho/\rho$ **~ 10^{-5} to** $\delta\rho/\rho$ **≤ 1** unity, worked out by Jeans (1910) and Lifshitz (1946).
  - But: We (humans) are overdense by a factor of 10^{28}!
  - Galaxies are overdense by a factor of 100 – 1000.
- We need to work out the rate of growth of $\delta\rho/\rho$ as a function of a(t) [←only depends on a(t)!]
- **To study the non-linear phase, we will look at**
  - Simple analytic approximations (Press-Schechter)
  - Numerical simulations

We start with the continuity equation and neglect radiation and any pressure forces for now:

$$\left(\frac{\partial \rho}{\partial t}\right)_p + \vec{\nabla}_p \left(\rho \, \vec{v}_p\right) = 0$$

and the equation of motion:

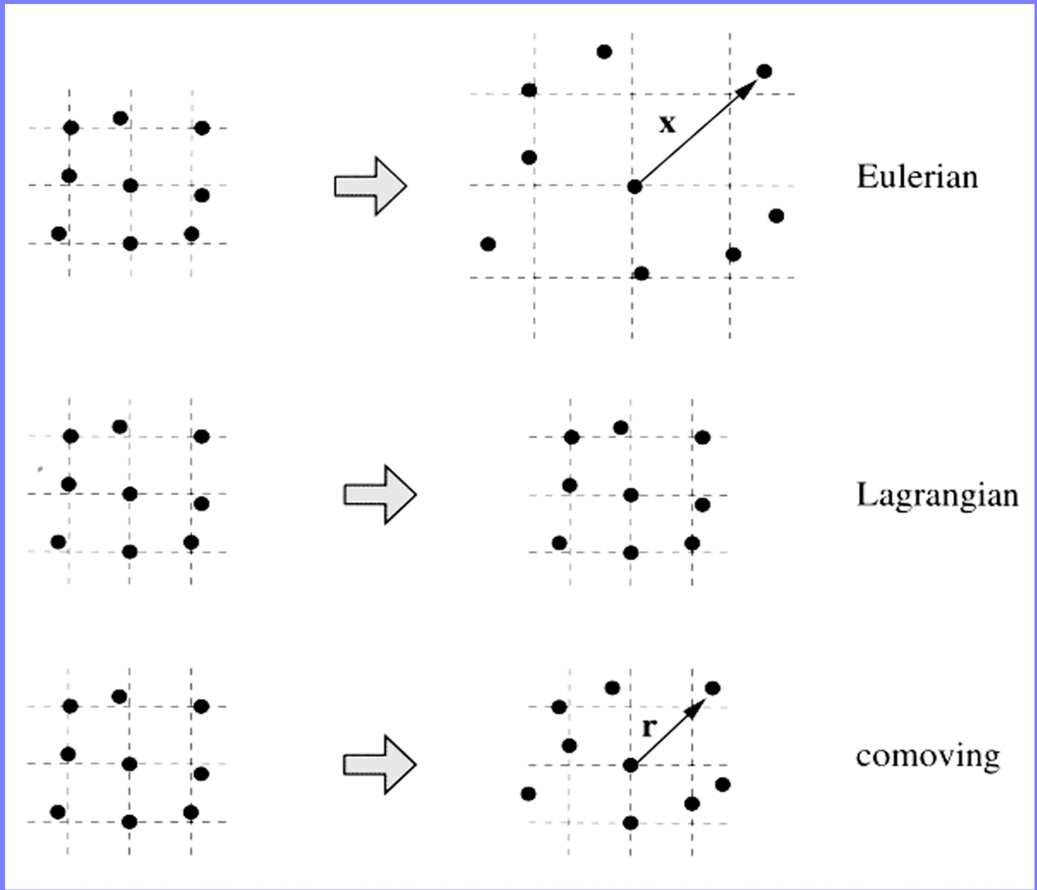
$$\left(\frac{\partial \vec{v}}{\partial t}\right)_p + \left(\vec{v}_p \cdot \vec{\nabla}_p\right)\vec{v}_p = -\frac{\vec{\nabla}_p P}{\rho} - \vec{\nabla}_p \Phi$$

∇_p is the derivative with respect to the proper (not co-moving) coordinate.

- In addition, we have Poisson's Equation:

$$\nabla_p^2 \Phi = 4\pi G\rho$$

- At this point, we have the choice of a co-ordinate system that simplifies the analysis.



- As the homogeneous, unperturbed universe is stationary in a coordinate frame that expands with the Hubble flow, we consider these equations in *co-moving coordinates*

  - in co-moving coordinate positions are constant and velocities are zero

$$\vec{x} \equiv \vec{r}_p(t) / a(t)$$

$\vec{x}$ = comoving position; r_p = proper position

$$\vec{v}_p = \dot{a}(t)\vec{x} + \vec{v}(\vec{x}, t)$$

v_p = proper velocity

$\vec{v}$ = comoving (peculiar) velocity = $a(t)\dot{\vec{x}}$

- Now we separate the uniform part of the density from the perturbation:

$$\rho = \bar{\rho}(t)\left[1 + \delta(\vec{x}, t)\right]$$

with $\bar{\rho} = \rho_0 / (1+z)^3$, accounting for the Hubble expansion

Note that : $$\frac{\dot{\bar{\rho}}}{\bar{\rho}} = -3\frac{\dot{a}}{a}$$

- To re-write the above equations, we need to explore how these derivatives differ between proper and co-moving coordinate systems:

a) temporal derivatives

$$\left(\frac{\partial f}{\partial t}\right)_{\text{proper}} = \left(\frac{\partial f}{\partial t}\right)_{\text{comov}} - \left(\frac{\dot{a}}{a}\right)\vec{x}\cdot\vec{\nabla} f$$

$\vec{\nabla} f$ taken in the co moving coordinates

b) spatial derivative $\vec{\nabla} = a(t)\vec{\nabla}_p$

- Apply this to the continuity equation (mass conservation):

$$\left(\frac{\partial}{\partial t} - \frac{\dot{a}}{a}\vec{x}\cdot\vec{\nabla}\right)\{\overline{\rho}(t)(1+\delta)\} + \frac{\overline{\rho}(t)}{a}\vec{\nabla}[(1+\delta)(\dot{a}\vec{x}+\vec{v})] = 0$$

- If we now use $\dot{\bar{\rho}} = -3\bar{\rho}\,\dot{a}/a$ and $\vec{\nabla}\bar{\rho}(t) = 0$

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla}\left[(1+\delta)\vec{v}\right] = 0$$

and assuming δ and $\vec{v}$ are small

$$\boxed{\frac{\partial \delta}{\partial t} + \frac{1}{a}\left(\vec{\nabla}\cdot\vec{v}\right) = 0}$$

this is a continuity equation for perturbations!

where $\delta(x) = \frac{\rho(x)}{\rho} - 1$ and $\vec{v}$ is the peculiar velocity

Define the *potential perturbation*, ϕ(x,t), through

$$\Phi(\vec{x},t)=\frac{4\pi}{3}G\overline{\rho}(t)a^2(t)\cdot x^2+\varphi(\vec{x},t)$$

$$\Rightarrow \boxed{\nabla^2\varphi = 4\pi G\overline{\rho}(t)a^2(t)\delta}$$ differs by a^2

perturbative Poisson's Equation in **co-moving** coordinates

Similar operations for the equation of motion
in co-moving coordinates!

$$\frac{\partial\vec{v}}{\partial t}+\frac{\dot{a}}{a}\vec{v}=-\frac{1}{a}\vec{\nabla}\varphi$$

<u>Note</u>: because velocities are assumed to be small, the

term $\frac{1}{a}(\vec{v}\cdot\vec{\nabla})\vec{v}$ has been dropped on the left.

As for the acoustic waves, these equations can be combined to:

$$\frac{\partial^2 \delta}{\partial t^2} + 2 \frac{\dot{a}}{a} \frac{\partial \delta}{\partial t} = 4\pi G \rho \delta$$

This equation describes the evolution of the fractional density contrast $\delta \equiv \delta\rho / \rho$ in an expanding universe!

Note:

- for da/dt=0 it is a wave/exponential growth equation (= „Jeans Instability")
- the expansion of the universe, $\dot{a}(t)$, acts as a damping term
- Note: this holds (in this simplified form) for any δ(x,t)

→ Mapping from early to late fluctuations = f(a(t))!

Simplest solutions:

(1) flat, matter dominated $\Omega_m \sim 1$ universe

$$\Longrightarrow a(t) \sim t^{2/3}$$

The Ansatz $\delta(\vec{x}, t) = A(\vec{x})t^{a} + B(\vec{x})t^{-b}$

a,b > 0 yields:

$$\delta(t) = At^{2/3} + Bt^{-1}$$

or

$$\delta = \frac{\delta\rho}{\rho} \sim a(t) \sim \frac{1}{1+z}$$

A = growing mode; B = decaying mode (uninteresting)

⇒ no exponential growth, but fractional fluctuations grow linearly with the overall expansion!

(2) low-density universe

$$\Omega_0 \longrightarrow 0 \Longrightarrow \overline{\rho} \longrightarrow 0 \quad \Longrightarrow \quad \delta(\vec{x}, t) = \delta(\vec{x})$$

constant with time, i.e. all perturbations are „frozen in"

(3) accelerating expansion (Cosmological constant)

Fractional density contrast decreases (in linear approximation)

⇒ all density perturbations grow, but at most proportional to

$\frac{1}{1+z}$ for $\Omega_{Mass} \leq 1$.

In the pressureless limit the growth rate is independent of the spatial structure.

Linear growth in an expanding universe: Simplest Version

- Growth rate independent of spatial scale, solely a function of a(t).

1) $\delta(z)=\delta(z=0)/D_{lin}(z)$ linear growth factor D_{lin}

2) $\delta \sim a(t) \sim 1/(1+z)$, or slower

- Complications:
  - Gas/radiation pressure
  - Causality, horizons
  - Non-linearity, baryons, …

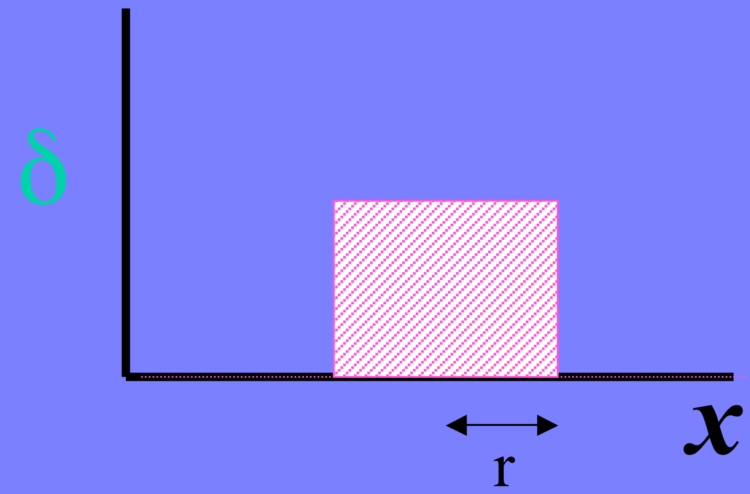
3.2. Structure growth beyond linear perturbations:

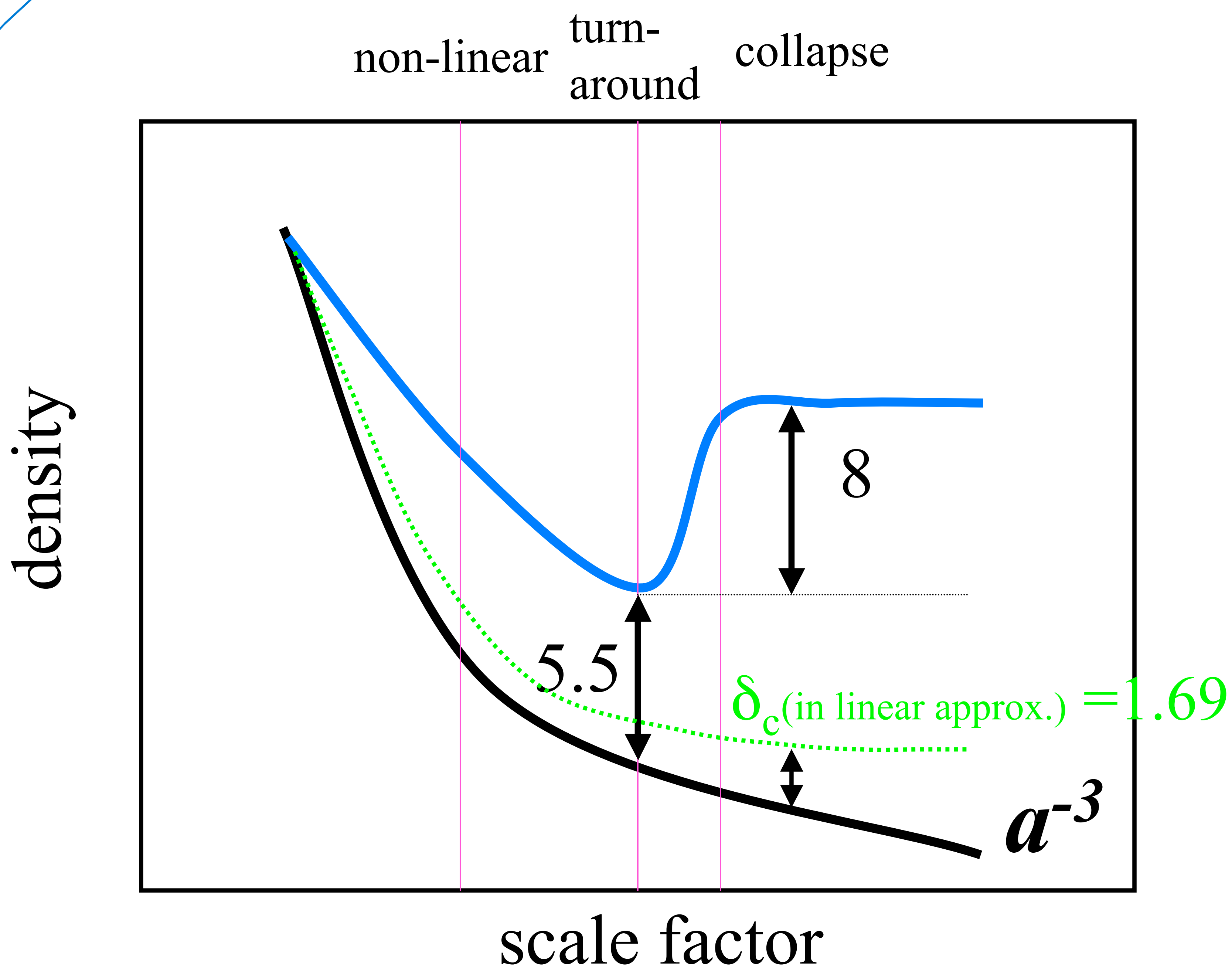
The 'top-hat model' (spherical collapse)

- consider a uniform, spherical perturbation

$$\delta_i = \rho(t_i)/\rho_b(t_i) - 1$$

$$M = \rho_b(4\pi r_i^3/3)(1+\delta_i)$$





non-linear
turn-around
collapse
density
8
5.5
δc(in linear approx.) =1.69
a-3
scale factor

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} + \frac{\Lambda}{3}r$$

$$\dot{r} = H_0\left[\frac{\Omega_0}{r}(1+\delta_i)\frac{r_i^3}{a_i^3} + \Omega_\Lambda r^2 - K\right]$$ = 0 at turnaround

$$t_{coll} = 2\int_0^{r_{ta}} \frac{dr}{\dot{r}}$$

Solution for collapsing top-hat (Ω_m=1)

- *turnaround* ($r=r_{max}$, $dr/dt=0$) occurs at $\delta_{lin}\sim 1.06$
- *collapse* ($r=0$): $\delta_{lin} \sim 1.69$
- *virialization*: occurs at $2t_{max}$, and $r_{vir} = r_{max}/2$

- where δ_{lin} is the 'linearly extrapolated overdensity'

- → we can use the simple linear theory to predict how many objects of mass M will have 'collapsed and virialized' at any given epoch

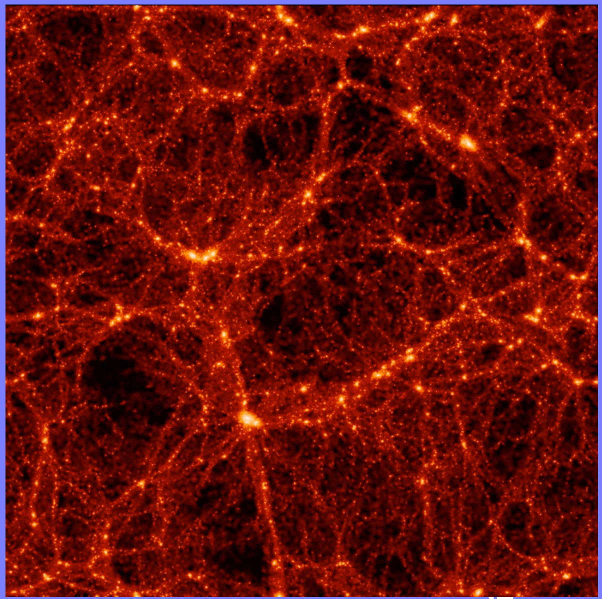
- How does mass enter? $\delta(\text{init}) = f(M)$

e.g. Padmanabhan p. 282

The halo mass function

- the halo mass function is the number density of collapsed, bound, virialized structures per unit mass, as a function of mass and redshift

→ dn/dM (M, z)



The Press-Schechter Model

- a generic prediction of inflation (supported by observations of the CMB) is that the primordial density field d is a Gaussian random field
- the variance is given by $S=\sigma^2(M)$, which evolves in the linear regime according to the function $D_{lin}(z)$
- at any given redshift, we can compute the probability of living in a place with $\delta>\delta_c$

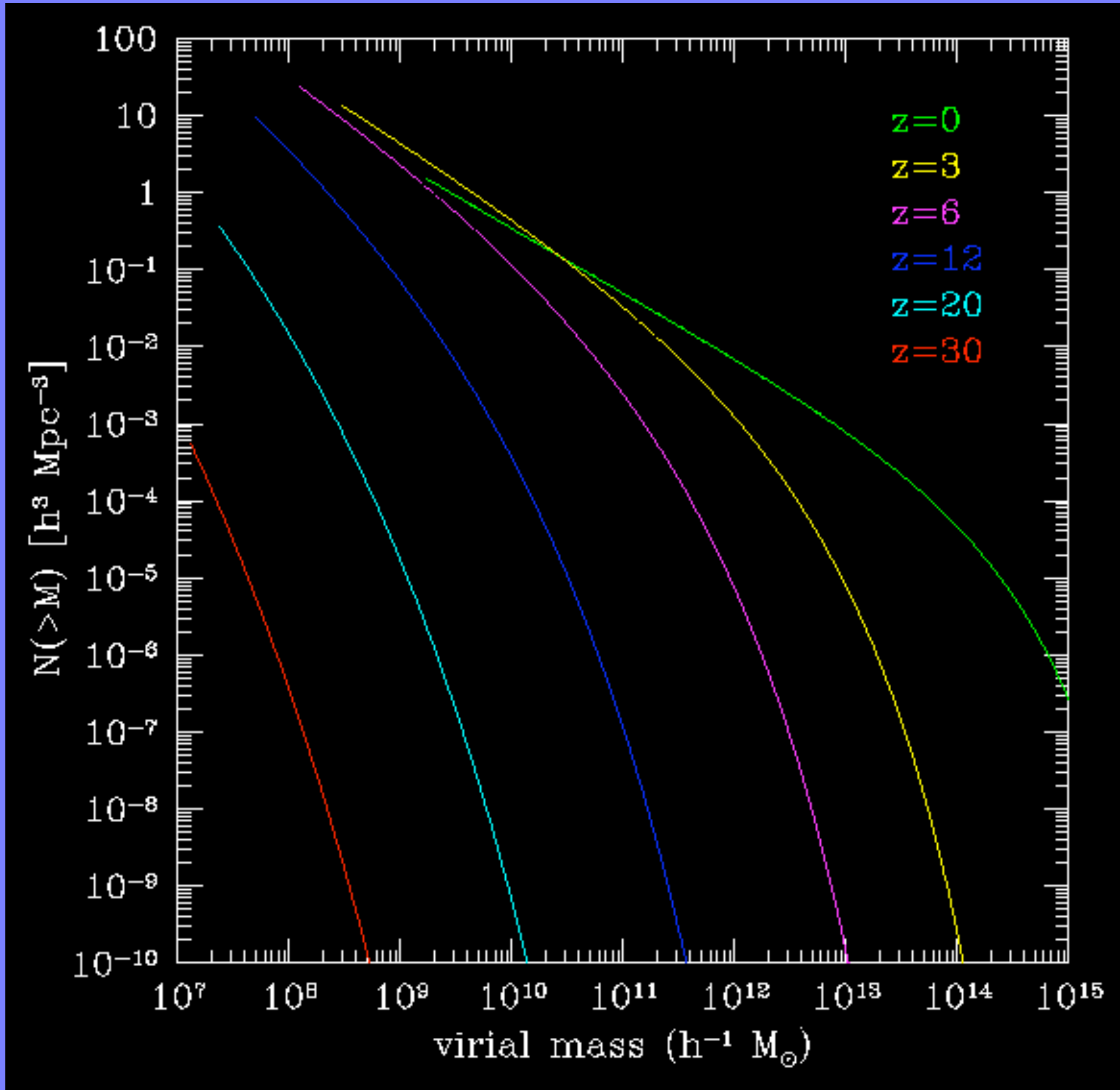
$$p(\delta>\delta_c \mid R) = ½ [1\text{-erf}(\delta_c/(2^{1/2}\, \sigma(R))]$$

number density of halos (halo mass function):

$$\frac{dn}{dM}(M,z)\,dM = \frac{\bar{\rho}_0}{M} f(S,\omega)\frac{d\sigma}{dM}dM$$

$$\frac{dn}{dM}(M,z)\,dM = \sqrt{\frac{2}{\pi}}\frac{\bar{\rho}_0}{M}\frac{\delta_c(t)}{\sigma^2(M)}\frac{d\sigma}{dM}\exp\left[-\frac{\delta_c^2(t)}{2\sigma^2(M)}\right]dM$$

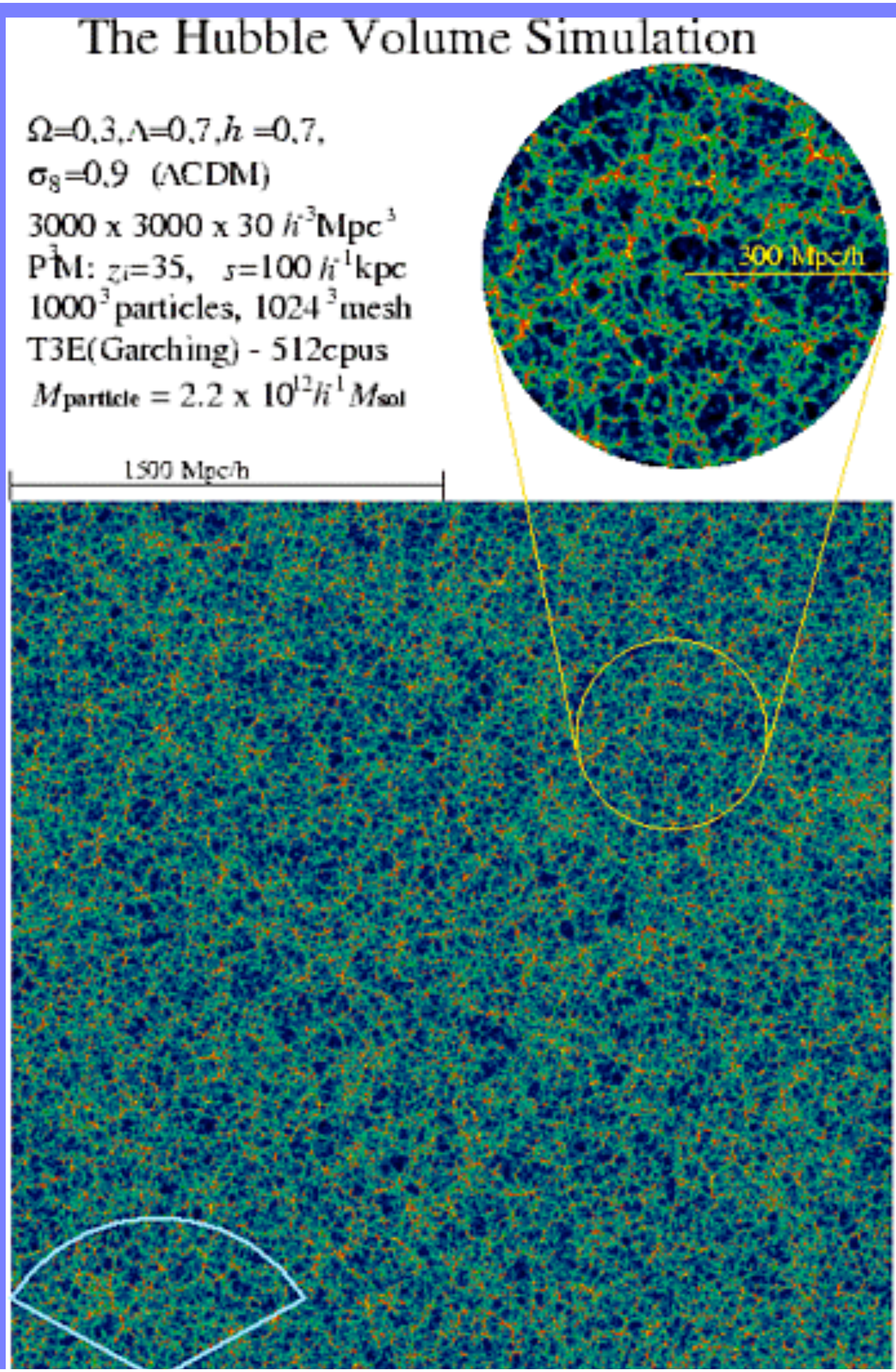
Resulting: cumulative halo mass function



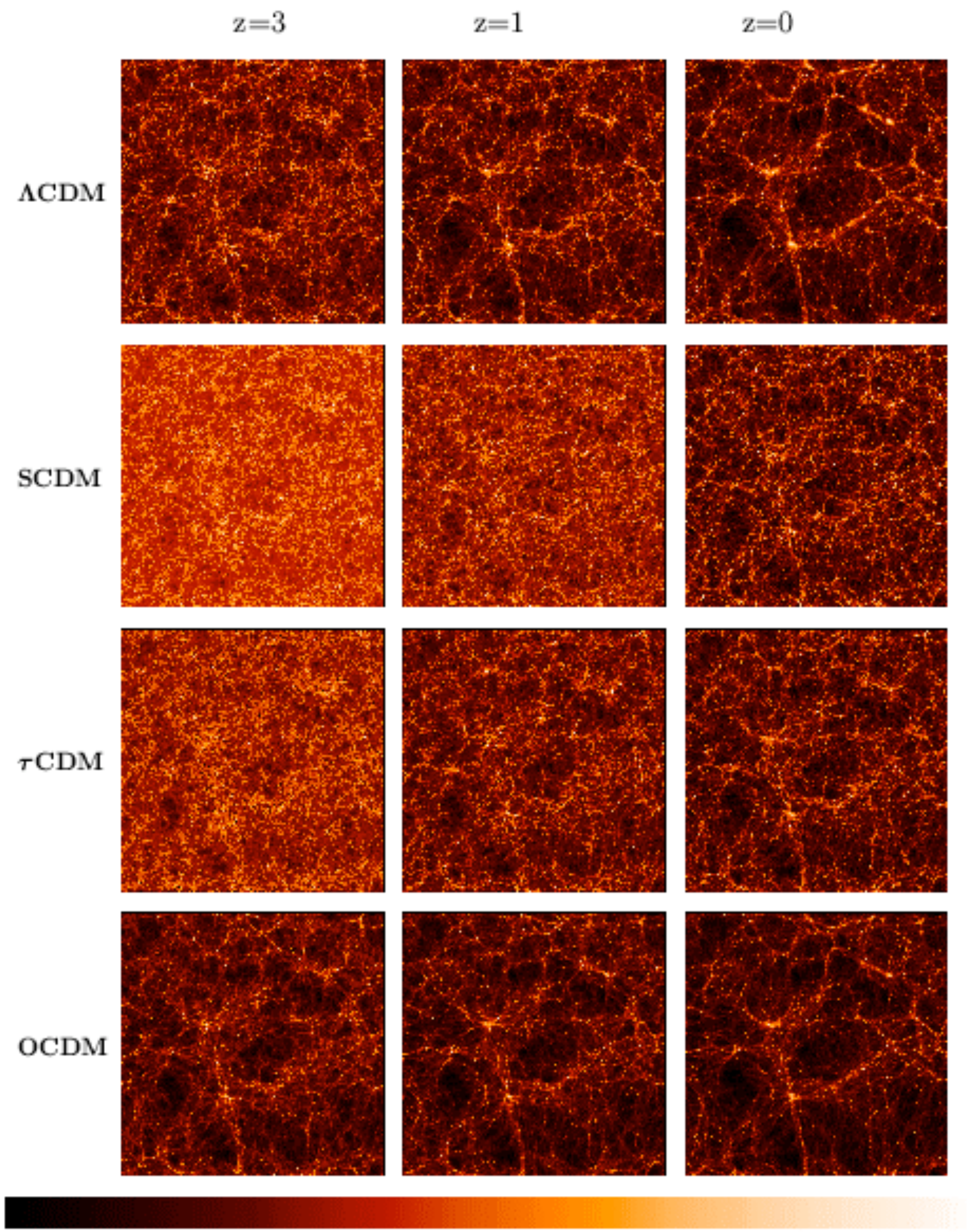
Numerical Calculations of Structure growth

(see also Numerical Cosmology Web-Pages at www.aip.de and www.mpa-garching.mpg.de

- Simulate (periodically extended) sub-cube of the universe.
- Gravity only (or include hydrodynamics)
  - Grid-based Poisson-solvers
  - Tree-Codes (N logN gravity solver)
- Up to 10^9 particles (typically 10^7)
- Need to specify
  - Background cosmology i.e. a(t),**r**
  - Initial fluctuation (inhomogeneity) spectrum
  - Assumption of "Gaussian" fluctuations


The Hubble Volume Simulation
Ω=0,3,Λ=0,7,h =0,7,
σ8=0,9 (ΛCDM)
3000 x 3000 x 30 h⁻³Mpc³
P³M: zi=35, s=100 h⁻¹kpc
1000³ particles, 1024³ mesh
T3E(Garching) - 512cpus
Mparticle = 2.2 x 10¹² h⁻¹ Msol
300 Mpc/h
1500 Mpc/h

Expansion History (=Mass Energy Density) Determines the Growth of Structure



The Mass Profiles of Dark Matter Halos in Simulations (Navarro, Frenk and White 1996/7)

$$\rho(r) = \frac{\delta_s}{(r/r_s)(1 + r/r_s)^2}$$

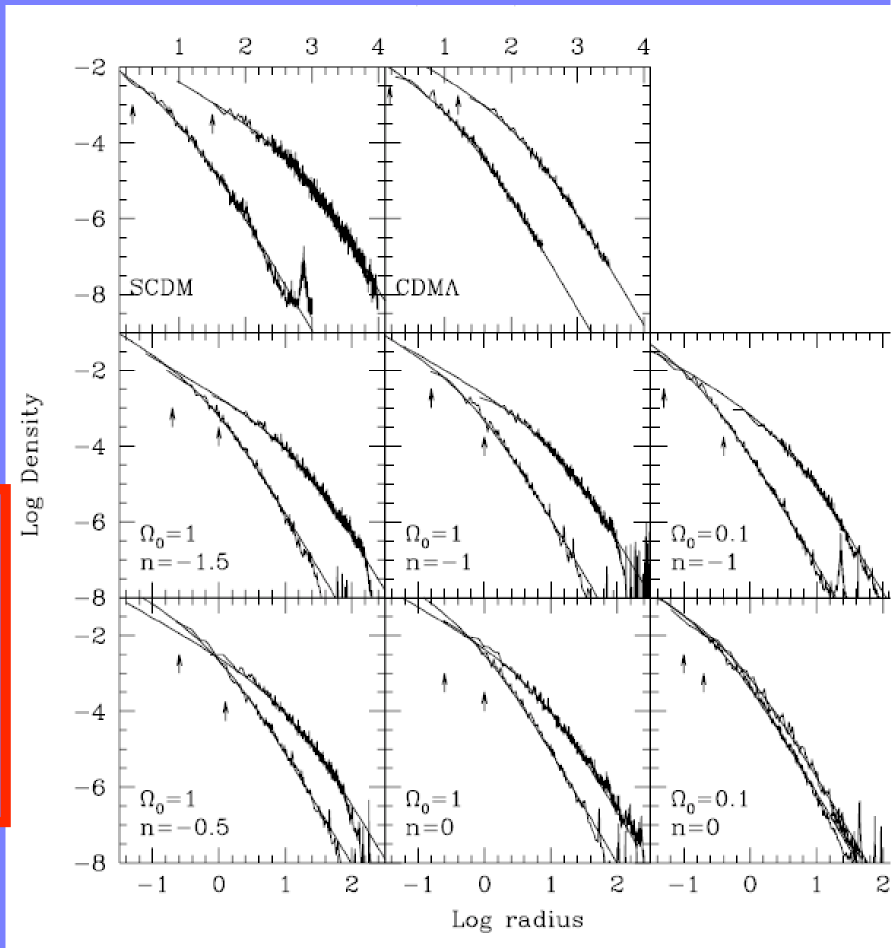
$$\delta_c = \frac{200}{3} \frac{c^3}{[\ln(1+c) - c/(1+c)]}$$

With c ~ r_{Vir}/r_s

The halo profiles for different masses and cosmologies have the same "universal" functional form:

$\rho \sim r^{-1}$ and $\rho \sim r^{-3}$ at small/large radii

Concentration is f(mass) → nearly 1 parameter sequence of DM halo mass profiles!



Summary

- The growth of (large scale) structure can be well predicted by
  - Linear theory
  - Press-Schechter (statistics of top-hat)
  - Numerical Simulations
- Density contrast does not grow faster than a(t) under gravity only.
- Several mechanisms can suppress growth
  - Pressure and accelerating expansion