1. Übung zu: Astronomie & Astrophysik II

(Due: Thu., 22.4.2004)

Problem 1.1: Cooling of White Dwarfs

White Dwarfs (WD) still radiate after ejection of their envelope. The required energy is taken largely from the thermal energy of the remaining ions – since the electrons of the star are already in a degenerate state, not much energy can be generated from contraction. You are asked to first compute the thermal energy of a WD and then conclude a potential remaining radiation time from this.

a) The heat capacity per ion c_i shall be given (assumed to be only a function of temperature)

$$U_i = \int c_i dT.$$

Assume $c_i = 3/2k$ for a non-degenerate monoatomic gas and derive a relation for the total heat capacity (or thermal energy) U for a WD of mass M, temperature T and composed only of ions of effective mass $A \cdot m_u$ with $A = 5$.

Answer: $U = 3/2kTM/(5m_u)$

b) How large is the heat capacity for $T = 10^7\,\mathrm{K}$ and $M = 1\,\mathrm{M}_\odot$?

Answer: $U = 5 \cdot 10^{40}\,\mathrm{J} = 5 \cdot 10^{47}\,erg$

c) How does this relate to the bolometric radiation of the sun?

Answer: $t = 5 \cdot 10^{47}\,\mathrm{erg}/3.86 \cdot 10^{33}\,\mathrm{erg/s} = 1.3 \cdot 10^{14}\,\mathrm{s} = 4.1 \cdot 10^6\,\mathrm{a}$

d) From the ongoing radiation the thermal energy of the WD continuously decreases. Write down the relation between the change in thermal energy $-dU/dt$ and the luminosity L of the WD. Assume, that the luminosity is proportional to $T^{3.5}$ and the mass, with a constant of proportionality $C = 2 \cdot 10^6\,\mathrm{erg\,s^{-1}\,M_\odot^{-1}}$.

Answer: $-\mathrm{d}U/\mathrm{d}t = -\mathrm{d}(3/2kTM/(5m_u))/\mathrm{d}t = C \cdot M \cdot T^{3.5}$

e) Determine the cooling time $\tau = (t - t_0)$ by integration, assume here that $T(t_0) \gg T(t)$ and express τ in terms of luminosity.

Answer: $3/5k/(5m_u) \cdot (T^{-5/2} - T_0^{-5/2}) = C(t - t_0) = C \cdot \tau \Leftrightarrow \tau = 3/5kTM/(5m_u L)$

f) How large is τ for $L = 10^{-3} L_\odot$, with T and M as in part c)? What does this mean for the possibility to actually observe WDs?

Answer: 1.65 Gyr, they are observable, if once in a while new ones are produced.

Problem 1.2: Neutrinos from SN1987a

With the end of fusion inside a star also the outward forces stop that counteract the gravitational force and keep the star stable. Stars with masses from $\sim$8 to $\sim$25 $\mathrm{M}_\odot$ subsequently collapse from the influence of gravitation down to a much smaller radius at which the degeneration of neutrons creates a new counteracting force. They become Neutron Stars. In this collapse a large part of the stellar mass is ejected in a supernova explosion, while large amounts of energy is released.

a) Assume the core of a massive star, core mass $M = 1.4\,\mathrm{M}_\odot$, collapses from a radius of $r_i = 1000$ km to a radius of $r_f = 10$ km. Show that the emitted energy is $\sim 3.1 \cdot 10^{53}$ erg, under the assumption that both before and after the collapse the density in all parts of the core is constant.

Answer: Gravitational energy for a sphere of homog. density ρ and radius R:

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\begin{aligned}
U &= -GmM/r \qquad \text{(testmass)} \\
\Rightarrow E_{\text{grav}} &= \int_{\text{sphere}} -GdV\rho M(r)/r \\
M(r) &= 4/3\pi r^3 \rho \\
dV &= r^2 \sin\theta d\theta d\phi \\
\Rightarrow E_{\text{grav}} &= -G4/3\pi\rho^2 \int_{\text{sphere}} r^4 \sin\theta d\theta d\phi dr \\
&= -G16/3\pi^2\rho^2 \int_0^R r^4 dr \\
&= -G16/15\pi^2\rho^2 R^5 = -G3/5M^2/R
\end{aligned}
$$

Now difference for two radii:

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\begin{aligned}
E_{\text{diff}} &= -G3/5M^2(1/R_i - 1/R_f) \\
&= -G3/5(1.4M_\odot)^2(1/10^6 - 1/10^4)\,\mathrm{m} \\
&= 3.14 \cdot 10^{50} \cdot 9.9^{-5}\,\mathrm{J} = 3.1 \cdot 10^{46}\,\mathrm{J} = 3.1 \cdot 10^{53}\,\mathrm{erg}
\end{aligned}
$$

b) $\sim$99% of the energy releasen in a supernova is emitted in the form of neutrinos that are created in process of converting normal elements to neutrons. The supernova SN 1987a is at a distance of $5.1 \cdot 10^4$ pc from earth. Compute, how many neutrinos from SN 1987a can be expected in experiments on earth. Assume as an energy the above $3.1 \cdot 10^{53}$ erg, as energy of the neutrinos $E_\nu = 20\,\mathrm{MeV}$, as detector surface $500\,\mathrm{m}^2$ and a detection probability of 10^{-16}.

Answer:

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\begin{aligned}
E &= 3.1 \cdot 10^{46}\,\mathrm{J} = 1.94 \cdot 10^{65}\,\mathrm{eV} = 9.68 \cdot 10^{57}\nu \\
r &= 5.1 \cdot 10^4\,\mathrm{pc} = 1.57 \cdot 10^{21}\,\mathrm{m} \\
A &= 3.1 \cdot 10^{43}\,\mathrm{m}^2 \\
n_{\text{event}} &= 9.68 \cdot 10^{57} \cdot 500/3.1 \cdot 10^{43} \cdot 10^{-16} \cdot 0.99 = 15.5
\end{aligned}
$$